A. Definition: Two fields are isomorphic if they are the same after renaming elements. Formally:

Fields $K$ and $L$ are isomorphic if there is a bijection $K \xrightarrow{\phi} L$ such that $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ for all $x, y \in K$. The map $\phi$ is called an isomorphism (or "renaming").

(1) Prove that an isomorphism of fields respects both the additive and the multiplicative identity elements: that is, $\phi(0_K) = 0_L$ and $\phi(1_K) = 1_L$.

(2) Prove that up to isomorphism, there is exactly one field with two elements.

(3) Prove that up to isomorphism, there is exactly one field with three elements.

B. Consider the field extension $L = \mathbb{Q}(\sqrt[3]{3}, \sqrt[3]{3}, \sqrt[3]{3})$ of $\mathbb{Q}$.

(1) Find $[L : \mathbb{Q}]$.

(2) What are the possible dimensions of $\mathbb{Q}$-vector subspaces $V$ of $L$?

(3) What are the possible dimensions of $\mathbb{Q}$-vector subspaces $V$ of $L$ if $V$ is also a subfield of $L$?

(4) Exhibit an example of a subfield $K$ of $L$ with each of the possible degrees you claimed in (3). You may describe $K$ by giving a set of field generators over $\mathbb{Q}$.

(5) Exhibit a $\mathbb{Q}$-vector subspace $V$ of $L$ that is not a subfield.

C. Let $K \hookrightarrow L$ be a field extension of prime degree $p$. Prove that for every $\alpha \in L \setminus K$, the minimal polynomial of $\alpha$ over $K$ has degree $p$. [Hint: Review lecture notes from 12.9.]

D. Find the minimal polynomial for $i + 1$ over $\mathbb{Q}$.

E. Let $K = \mathbb{Q}(S)$ where $S$ is the set of all the complex $p$-th roots of unity, where $p$ is prime. Prove that $[K : \mathbb{Q}] = p - 1$. You may assume that the polynomial $x^p - 1 + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible.\(^1\)

F. Find a generator for the (principal) ideal in $\mathbb{Q}[x]$ generated by the elements $x^6 - 4x^2$ and $x^3 - 2x$.

G. Let $\mathbb{F}_2$ be the field of two elements (call them 0 and 1).

(1) Prove that $x^2 + x + 1$ is an irreducible polynomial in $\mathbb{F}_2[x]$.

(2) How many elements are in the quotient ring $R = \mathbb{F}_2[x]/(x^2 + x + 1)$? List them explicitly.

(3) Make addition and multiplication tables for $R$.

(4) Explain why $R$ a field.

Bonus. Recall the following Definitions: Let $K$ be any field such that $\mathbb{Q} \subset K \subset \mathbb{R}$. A point $p = (x_1, y_1)$ in the Cartesian plane is $K$-rational if $x_1, y_1 \in K$. A line is $K$-rational if it is determined by two $K$-rational points. A circle is $K$-rational if its center is $K$-rational and it passes through a $K$-rational point.

(1) Prove that the intersection of a $K$-rational line with a $K$-rational circle will consist of (at most) two $L$-rational points, where $L$ is a quadratic extension of $K$.

(2) Prove that the intersection of two $K$-rational circles will consist of (at most) two $L$-rational points, where $L$ is a quadratic extension of $K$.

\(^1\)meaning, after renaming elements

\(^2\)It is hard, in general, to test irreducibility of polynomials, just as it is hard to know whether large integers are prime or not. We will soon be able to prove that this polynomial is irreducible, however, using a trick called Eisenstein’s criterion.