A. Prove that if $p$ is prime, then $\sqrt[n]{p}$ is not constructible for $n \geq 3$, unless $n$ is a power of 2.

B. Let $\theta$ be an angle $0 \leq \theta \leq 2\pi$.

(1) Show that the angle $\theta$ can be constructed (using only straightedge and compass) if and only if the real number $\cos \theta$ is constructible.

(2) Prove that $\cos \theta$ is constructible if and only if $\sin \theta$ is constructible. [See if you can two different proofs: one using the definition of constructibility and one using that $K$ is quadratically closed.]

C. Let $f(x) = ax + bx + cx^2 + dx^3 \in \mathbb{Z}[x]$ be a polynomial of degree 3. For a prime $p$, let $\overline{f}(x) = \overline{a}x + \overline{b}x + \overline{c}x^2 + \overline{d}x^3 \in \mathbb{Z}_p[x]$ be the polynomial obtained by reducing each coefficient modulo $p$.

(1) Show that if a prime $p$ does not divide $d$, then the degree 3 polynomial $f$ is irreducible if $\overline{f}(\overline{a}) \neq 0$ for any $n = 0, 1, \ldots, p-1$.

(2) Show that for $n, m \in \mathbb{Z}$, the polynomial $1 + (x+n)(x+n+1)(x+m)$ is irreducible over $\mathbb{Z}$.

D. More tricks for checking irreducibility:

(1) Show that a polynomial $f(x) \in K[x]$ is irreducible if and only if for some (equivalently, every) non-zero $k \in K$ the polynomial $f(kx)$ is irreducible.

(2) Show that the polynomial $4x^3 - 3x - \frac{1}{2}$ is irreducible over $\mathbb{Q}$ by applying the previous statement with $k = \frac{1}{2}$, and then using the technique from Problem C last week.

(3) Show that the polynomial $4x^3 - 3x - \frac{1}{2}$ is irreducible over $\mathbb{Q}$ by using reduction mod 5. [Hint: See Problem E from last week and Lecture from Sept 28.]

(4) Explain why an angle of measure $\pi/9$ can not be constructed. [Hint: See Lecture from Sept 28.]

E. Regular polygons.

(1) Given a regular polygon with $n$ sides, explain how to construct a regular polygon with $2n$ sides using only straightedge and compass.

(2) Prove that regular 18-gon is not constructible, and therefore nor is an 9-gon.

(3) It is a hard question, answered by Gauss, to determine exactly which regular polygons are constructible. Look up the answer on line.

F. Let $L$ be an an extension of a field $K$, and let $\alpha \in L$. Let $\nu : K[x] \to K(\alpha)$ be the ring homomorphism sending each polynomial to $f(x)$ to $f(\alpha)$

(1) Show that $\nu$ is injective if and only if $\alpha$ is transcendental.

(2) Show that if $\alpha$ is algebraic, then $\ker \nu$ is generated by the minimal polynomial $g(x)$ of $\alpha$.

(3) Show that if $\alpha$ is algebraic if and only if $\nu$ induces an isomorphism $K(x) \to K(\alpha)$ for some polynomial $g \in K[x]$. 

(4) Show that if $\beta \in L$ is another root of the minimal polynomial $g(x)$, then there is an isomorphism $\phi : K(\beta) \to K(\alpha)$ which fixes $K$. 