Algebra 2: Harjoitukset 5.

A. Let \( A \) be the set of all complex numbers that are algebraic over \( \mathbb{Q} \).

1. Let \( \alpha \) and \( \beta \) be elements of \( A \) of degrees \( N \) and \( M \) over \( \mathbb{Q} \), respectively (meaning that their minimal polynomials are of degrees \( N \) and \( M \)). Show that \( \mathbb{Q}(\alpha, \beta) \) has degree at most \( NM \) over \( \mathbb{Q} \).

2. Show that \( A \) is a subfield of \( \mathbb{C} \). What is \( |A : \mathbb{Q}| \)?

3. Show the set of a real algebraic numbers (meaning real numbers which are algebraic over \( \mathbb{Q} \)) is a subfield of \( \mathbb{R} \).

4. Show that the field of constructible numbers is a subfield of the field of real algebraic numbers.

B. Definition: Fix a subfield \( K \subset \mathbb{C} \). Let \( f(x) \in K[X] \) be any polynomial. The splitting field of \( f \) is the field \( K(\alpha_1, \ldots, \alpha_n) \) where \( \{\alpha_1, \ldots, \alpha_n\} \) is the complete set of complex roots of \( f \).

1. Compute\(^1\) the splitting field of \( x^2 + 1 \in \mathbb{Q}[x] \), and its degree over \( \mathbb{Q} \).

2. Compute the splitting field of \( x^3 - 1 \in \mathbb{Q}[x] \), and its degree over \( \mathbb{Q} \).

3. Compute the splitting field of \( x^4 - 1 \in \mathbb{Q}[x] \), and its degree over \( \mathbb{Q} \).

4. Compute the splitting field of \( x^5 - 1 \in \mathbb{Q}[x] \), and its degree over \( \mathbb{Q} \).

5. Compute the splitting field of \( x^3 - 3 \in \mathbb{Q}[x] \), and its degree over \( \mathbb{Q} \).

C. Let \( f = ax^2 + bx + c \in K[x] \) (where \( K \subset \mathbb{C} \)). Recall that the discriminant of \( f \) is the quantity \( b^2 - 4ac \).

1. Let \( L \) be the splitting field of a quadratic polynomial in \( f(x) \in \mathbb{Q}[x] \). Show that \( |L : \mathbb{Q}| \leq 2 \), with equality if and only if the discriminant of \( f \) is not a square in \( \mathbb{Q} \).

2. Show that two irreducible quadratic polynomials over \( \mathbb{Q} \) have the same splitting field if and only if the ratio of their discriminants is a square in \( \mathbb{Q} \).

D. Let \( K \subset L \) be subfields of \( \mathbb{C} \). Explain why \( K \) is an extension of \( \mathbb{Q} \). Explain why every automorphism of \( K \) is a \( \mathbb{Q} \)-automorphism. Explain why there are natural inclusions of the groups \( \text{Gal}(L/K) \subset \text{Gal}(L/\mathbb{Q}) \).

E. Find generators and relations for the Galois group of \( \mathbb{Q}(i, \sqrt{11}) \) over \( \mathbb{Q} \). What is its order?

F. Consider the set \( \mathbb{C} \) of complex numbers. Note that \( \mathbb{C} \) has many structures: for example, we can consider \( \mathbb{C} \) as a set, as a topological space, as a real manifold, as a complex manifold, as an abelian group, as a real vector space, as a complex vector space, as a rational vector space, as a field, as an extension field of \( \mathbb{R} \), or as an extension field of \( \mathbb{Q} \) (to name a few). Each of these different “categories” is defined by some additional structure on \( \mathbb{C} \). For each\(^2\) of these 11 categories, define the corresponding automorphism group of \( \mathbb{C} \) that respects the structure (for example, the final one listed would have automorphism group \( \text{Gal}(\mathbb{C}/\mathbb{Q}) \)). Construct a diagram showing the web of natural inclusions or maps among these groups. Where possible, describe the group as explicitly as you can. Consider whether any of your maps between groups are isomorphisms, and when not, give an explicit example of why not.

\(^1\)meaning, find generators over \( \mathbb{Q} \)

\(^2\)If you are a beginner, you may not recognize all these terms; just do what you can. You can look up definitions of manifolds for example, or just skip that part if they are unfamiliar.