Algebra 2: Harjoitukset 7.

A. Recall that if \( f(x) \in \mathbb{Q}[x] \) is any polynomial, with roots \( \beta_1, \ldots, \beta_n \in \mathbb{C} \), then the Galois group of \( f \) can be identified with a subgroup of \( S_n \). For each polynomial below, fix an ordering of the roots, and then write out the elements of the Galois group of \( f \) explicitly as permutations, using cycle notation.

\( (1) \) For \( x^5 - 1 \in \mathbb{Q}[x] \), express the elements of \( G \) in cycle notation in \( S_5 \).
\( (2) \) For \( x^4 - 5x^2 + 6 \in \mathbb{Q}[x] \), express the elements of \( G \) in cycle notation in \( S_4 \).
\( (3) \) For \( x^5 - 7 \in \mathbb{Q}[x] \), express the elements of \( G \) in cycle notation in \( S_5 \).

B. **Definition:** A subgroup \( H \) of a group \( G \) is **normal** if for all \( g \in G \) and all \( h \in H \), we have \( g^{-1}hg \in H \).

(1) Show that in an abelian group, every subgroup is normal.
(2) Show that the kernel of a group homomorphism is a normal subgroup.
(3) Let \( G/H \) denote the set of cosets \( gH \) with respect to \( H \). Prove that if \( H \) is normal, then there is a well-defined binary operation on \( G/H \)
\[
g_1H \circ g_2H = (g_1g_2)H
\]
making \( G/H \) into a group. [Hint: The issue is that there is more than one way to write a coset: \( gH = g'H \) for any \( g \in g'H \). We need to make sure the product is independent of the choice of representative for the coset.]
(4) Show that if \( H \) is normal, then the natural map \( G \to G/H \) taking each \( g \) to the corresponding coset \( gH \) is a group homomorphism with kernel \( H \).
(5) True or False: Let \( H \) be a subgroup of \( G \). Then \( H \) is normal if and only if it is the kernel of some homomorphism \( G \to G' \).

C. List all subgroup of \( S_5 \). Which are normal? For each normal subgroup \( H \), compute \( S_5/H \).

D. **Definition:** A group \( G \) acts on a set \( X \) if there is a map \( G \times X \to X \) sending \( (g, x) \mapsto g \cdot x \) such that

\( (1) \) \( g \cdot (h \cdot x) = (gh) \cdot x \) for all \( g, h \in G \) and all \( x \in X \).
\( (2) \) \( e \cdot x = x \) for all \( x \in X \).

i. Prove that the set \( G_x = \{ g \in G \mid g \cdot x = x \} \) is a subgroup of \( G \) (this is called the **stabilizer subgroup** of \( x \)).
ii. The orbit of \( x \in X \) is the set \( G \cdot x = \{ g \cdot x \mid g \in G \} \). Prove that \( G \cdot x = G \cdot y \) if and only if there exists \( h \in G \) such that \( hx = y \).
iii. Prove that for all \( x, y \in X \), \( G \cdot x = G \cdot y \) OR \( G \cdot x \cap G \cdot y = \emptyset \).
iv. Prove that if \( G \cdot x = G \cdot y \), then the stabilizers of \( x \) and \( y \) are conjugate— that is, there exists \( h \in G \) such that \( h^{-1}G_yh = G_x \).
v. Show that all elements in the orbit \( G \cdot x \) have the same stabilizer \( H \) if and only if the stabilizer \( H \) of \( x \) is a normal subgroup of \( G \).

E. Let \( G \) be the Galois group of a finite normal extension \( L/K \). Let \( \mathcal{F} \) be the set of all intermediate fields of \( L/K \).

\( (1) \) Show that for \( F \in \mathcal{F} \) and \( \phi \in G \), \( \phi(F) \) is a field.
\( (2) \) Find a natural (non-trivial) action of \( G \) on the set \( \mathcal{F} \) of intermediate fields of \( L/K \).
\( (3) \) Fix \( F \in \mathcal{F} \). Let \( H \) be the stabilizer of \( F \) under the action of \( G \). Find a natural group homomorphism \( H \to \text{Gal}(F/K) \) whose kernel is \( \text{Gal}(L/F) \).

F. Let \( L \) be the splitting field of \( x^3 - 3 \) over \( \mathbb{Q} \), and let \( G \) be the Galois group. [See Lecture from Oct 17].

\( (1) \) Describe each orbit of the action of \( G \) on \( \mathcal{F} \) (defined in E) explicitly (ie, list out the fields \( F \) in each orbit).
\( (2) \) For each \( F \in \mathcal{F} \) compute the stabilizer of \( F \) explicitly, by listing the elements of \( G \) in the stabilizer.
\( (3) \) Which intermediate fields \( F \) are normal extensions of \( K \)?
\( (4) \) Which \( F \in \mathcal{F} \) have normal stabilizer?