Let $f(x) = f_1(x)f_2(x) \in \mathbb{Q}[x]$, where $f_1$ and $f_2$ are irreducible polynomials of degree at least 2, whose splitting fields $K_1$ and $K_2$ are two different subfield of $\mathbb{C}$, neither contained in the other. Let $G$ be the Galois group of $f$ over $\mathbb{Q}$. Prove that $G$ has at least two different proper non-trival normal subgroups.

Suppose a finite group $G$ acts on a set $X$. Fix $x \in X$ and let $H \subset G$ be its stabilizer. Prove $|G| = |H||G \cdot x|$ where $G \cdot x$ is the orbit of $x$ under $G$. [Hint: Try to find a bijection from the set of cosets $G/H$ to the orbit.]

Let $G$ be the rotational symmetry group of the regular tetrahedron $X$, a solid with four faces consisting of equilateral triangles, four vertices and six edges. Consider the natural action of $G$ on $X$ by symmetries.

(1) Explain why there is a natural action of $G$ on the vertices of $X$. Compute the cardinalities of the orbit and stabilizer of each vertex. Use this (and B) to compute $|G|$.

(2) Explain why there is a natural action of $G$ on the faces of $X$. Compute the cardinalities of the orbit and stabilizer of each face. Use this (and B) to compute $|G|$ a different way.

(3) Explain why there is a natural action of $G$ on the edges of $X$. Compute the cardinalities of the orbit and stabilizer of each vertex.

(4) Find four non-equal subgroups of $G$ that are conjugate to each other by Exercise D(v) from last week. Is any of these normal in $G$?

Let $G$ be the rotational symmetry group of a sulfur hexafluoride molecule, a solid with four faces consisting of equilateral triangles, four vertices and six edges. Compute the order of the rotational symmetry group of a sulfur hexafluoride molecule. What about a Dodecaborane molecule? (There are nice pictures of these molecules on their respective wikipedia pages.)