A. Prove or disprove: Consider the field extension \( \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt[2]{7}, \sqrt[2]{5}, \sqrt[2]{11}) = L \). Then there exists an intermediate field \( K \) such that \( \mathbb{Q} \subset K \subset L \) with \([K : \mathbb{Q}] = 3\).

FALSE!

Consider the tower of field extensions \( \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt[2]{7}) \hookrightarrow \mathbb{Q}(\sqrt[2]{7}, \sqrt[2]{5}) \hookrightarrow \mathbb{Q}(\sqrt[2]{7}, \sqrt[2]{5}, \sqrt[2]{11}) = L \).

Each successive extension is quadratic (since we are adjoining square roots), hence of degree 2. [This assumes each successive square root we adjoin is not already in the previous field which strictly speaking requires some argument. But it doesn’t matter, since otherwise that extension would be of degree 1.] By the theorem on the multiplicativity of degree, the extension \( \mathbb{Q} \hookrightarrow L \) has degree \( 2^3 = 8 \) [or 2 or 2^2 if we don’t want to justify that each step is a proper extension]. So again by the multiplicativity of degree, any intermediate field extension must have degree dividing \( 2^3 \). So there is no intermediate extension of degree 3.

B. True or false. Explain. You may quote results from the homework or lecture.

EVERY FINITE FIELD HAS \( p \) ELEMENTS, FOR SOME PRIME NUMBER \( p \).

FALSE! We saw in the homework that there is a field with 4 elements, namely \( \mathbb{F}_2[x]/(x^2 + x + 1) \).

C. True or false. Explain. You may quote results from the homework or lecture.

THE IDEAL OF POLYNOMIALS IN \( \mathbb{C}[x] \) WHICH VANISH AT BOTH 0 AND 1 CAN NOT BE GENERATED BY ONE ELEMENT.

FALSE! We proved in the lecture that every ideal in \( \mathbb{C}[x] \) is generated by one element.