TRUE OR FALSE. Explain.

1. If complex numbers \( \alpha \) and \( \beta \) are non-zero algebraic numbers over \( \mathbb{Q} \), then so is \( \alpha^3 + \beta^7 \alpha + \frac{1}{\beta^2} \).

   TRUE. The first problem of the homework showed that the set of algebraic numbers forms a field. So if \( \alpha \) and \( \beta \) are algebraic, so is their sum, product, and any expression we can form from them using \(+, -, \cdot, :\). That is, every rational expression in \( \alpha \) and \( \beta \) (over \( \mathbb{Q} \)) is also algebraic (over \( \mathbb{Q} \)).

2. If \( L \) is the splitting field of a degree \( n \) polynomial \( f(x) \in \mathbb{Q}[x] \), then \( [L : \mathbb{Q}] = n \).

   False! Let \( f(x) = (x - 1)^n \) for \( n > 1 \). This is degree \( n \) but its splitting field is \( \mathbb{Q} \). Of course \( [\mathbb{Q} : \mathbb{Q}] = 1 \).

3. For \( \alpha = 17 + 2\sqrt[3]{7} \), we have \( \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt[3]{7}) \).

   TRUE: \( \alpha \in \mathbb{Q}(\sqrt[3]{7}) \), so \( \mathbb{Q}(\alpha) \subset \mathbb{Q}(\sqrt[3]{7}) \). But also \( \sqrt[3]{7} = \frac{3}{2}(\alpha - 17) \), so \( \sqrt[3]{7} \in \mathbb{Q}(\alpha) \) and \( \mathbb{Q}(\sqrt[3]{7}) \subset \mathbb{Q}(\alpha) \).

4. Let \( L_2 \) and \( L_3 \) be two field extensions of some field \( K \). If \( Gal(L_1/K) \cong Gal(L_2/K) \), then \( L_1 \cong L_2 \).

   False! Let \( K = \mathbb{Q}, L_1 = \mathbb{Q}(\sqrt[3]{2}), L_2 = \mathbb{Q}(\sqrt[3]{3}) \). Both Galois groups are cyclic of order two, hence isomorphic. But the fields are not isomorphic. If they were, then say \( \phi : L_1 \rightarrow L_2 \) is an isomorphism. Since it sends 1 to 1 and preserves the field structure, it must fix \( \mathbb{Q} \). Where can it send \( \sqrt[3]{2} \)? Suppose \( \phi(\sqrt[3]{2}) = a + b\sqrt[3]{3} \). Since \( 2 = (\sqrt[3]{2})^2 \), and \( \phi \) preserves the field structure, we have \( 2 = \phi(2) = \phi((\sqrt[3]{2})^2) = [\phi(\sqrt[3]{2})]^2 = x^2 \). This means that \( x = a + b\sqrt[3]{3} \in \mathbb{Q}(\sqrt[3]{3}) \) satisfies \( x^2 = (a + b\sqrt[3]{3})^2 = a^2 + 3b^2 + 2ab\sqrt[3]{3} = 2 \). This means that \( 2ab = 0 \) and \( a^2 + 3b^2 = 2 \). It is easy to see there are no such \( a, b \in \mathbb{Q} \) (since \( 2ab = 0 \) implies either \( a \) or \( b \) is zero, in which case either \( a^2 \) or \( 3b^2 \) is 2, which is impossible because both \( \sqrt[3]{2} \) and \( \frac{a}{b} \) are irrational).