Worksheet on Support and Associated Primes of Modules

Let $R$ be a commutative ring with 1. Let $M$ be an $R$ module.

**DEFINITION.** The **support** of $M$ is the subset $\{P \in \text{Spec } R \mid M_P \neq 0\} \subset \text{Spec } R$.

**PROPOSITION.** If $M$ is a finitely generated $R$-module, then $\text{Supp}(M)$ is the closed set of $\text{Spec } R$ given by $\mathbb{V}(\text{ann}_R(M))$ where $\text{ann}_R M$ is the ideal $\{r \in R \mid rM = 0\}$.

**DEFINITION.** A prime $P \in \text{Spec } R$ is an **associated prime** of $M$ if and only if there is an injective $R$-module map $R/P \rightarrow M$. The set of all associated primes of $M$ is called the **assassinator** of $M$ and denoted $\text{Ass}(M)$.

**THEOREM 1.** The set of associated primes of a non-zero finitely generated module over a Noetherian ring is non-empty and finite. That is, $0 < |\text{Ass}(M)| < \infty$.

**DEFINITION.** A **zero-divisor** on $M$ is an element $r \in R$ such that $rm = 0$ for some $m \in M \setminus \{0\}$.

**THEOREM 2.** Let $M$ be a finitely generated module over a Noetherian ring. The set of all zero-divisors on $M$ is the union of the Associated primes of $M$.

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(1) Let $M$ be an arbitrary $R$-module over an arbitrary ring $R$.
   (a) Show the support $M$ is empty if and only if $M = 0$. [Hint: Remember the worksheet on localization!]
   (b) Show that if $P \in \text{Supp}(M)$, then $\mathbb{V}(P) \subset \text{Supp}(M)$. [Hint: If $Q \supset P$, describe a natural map $M_Q \rightarrow M_P$.]
   (c) Show that the support of $R/I$ is $\mathbb{V}(I) \subset \text{Spec } R$.

(2) Consider the $\mathbb{Z}$-module $M = \bigoplus_{p \text{ odd prime}} \mathbb{Z}/p\mathbb{Z}$. Find the support of $M$ and prove it is not closed in $\text{Spec } \mathbb{Z}$. Why doesn’t this contradict the Proposition? [Hint: Remember $\otimes$ distributes over $\oplus$.]

(3) **PROOF OF THE PROPOSITION.** Let $M$ be a finitely generated module over an arbitrary $R$.
   (a) Show that $\text{ann}_R M$ is an ideal of $R$.
   (b) Show that if $m_1, \ldots, m_n$ generate $M$, then $\text{ann}_R(M) = \bigcap_{i=1}^n \text{ann}_R(m_i)$.
   (c) Prove $\text{Supp}(M) = \mathbb{V}(\text{ann}_R(M))$.

(4) Let $M$ be an arbitrary $R$-module over an arbitrary ring $R$. Fix $P \in \text{Spec } R$.
   (a) Show that $\in \text{Ass}(M)$ if and only if $P = \text{ann}_R x$ for some non-zero $x \in M$.
   (b) Show that if $R$ is a domain, the only associated prime of $R$ is $(0)$.
   (c) Let $R = K[x,y]$ and let $M = R/(xy, x^2)$. Show that $\{(x), (x,y)\} \subset \text{Ass}(M)$. [Hint: Use (a). It might also be useful to remember that $K[x,y]$ is a UFD.]

(5) Show that $\text{Ass}(M) \subset \text{Supp } M$ for any $M$. [Hint: Use the fact that $R_P$ is a flat $R$-module, so it preserves injections.]
(6) Let \( R = K[x, y] \). Fix any maximal ideal \( m \) such that \( R/m \cong K \). Let \( M = \text{Hom}_K(R, R/m) \).
   (a) Describe a natural \( R \)-module structure on \( M \).
   (b) Show that the \( R \)-linear map \( R \to M \) sending \( r \) to the composition \( R \xrightarrow{r} R \to R/m \)
       induces an embedding \( R/m \hookrightarrow M \).
   (c) Show that \( m \) is an associated prime of \( \text{Hom}_K(R, K) \).

(7) Let \( R = K[x_1, x_2, x_3, \ldots]/J \) where \( J = \langle x_t^{t+1} \mid t \in \mathbb{N} \rangle \). Prove that \( \text{Spec} \) \( R \) consists of one point and that \( \text{Ass} \) \( R \) is empty. Why doesn’t this contradict Theorem 1? Is the reverse inclusion in Problem (5) true?

(8) A USEFUL LEMMA. Let \( R \) be Noetherian ring and \( M \) a non-zero \( R \)-module. Show that the set of ideals \( \{ J \subset R \mid \exists m \in M \setminus \{0\} \text{ s.t. } J = \text{ann}_R(m) \} \) has a maximal element, and any such maximal element is prime. \([\text{Hint: If } xy \in \text{ann}_R m, \text{ consider } \text{ann}_R(xm).] \)

(9) PROOF OF THEOREM 2. Let \( M \) be a finitely generated module over a Noetherian ring.
   (a) Let \( P \in \text{Ass}_R(M) \). Prove every element of \( P \) is a zero-divisor on \( M \). \([\text{Hint: Use (4a).}] \)
   (b) Assume that \( rm = 0 \) for some non-zero \( m \in M \). Show that there exists \( s \in S \) such that \( \text{ann}_R(sm) \) is prime and contains \( r \). \([\text{Hint: Use ideas from (8).}] \)
   (c) Prove Theorem 2.

(10) PRIME CYCLIC FILTRATIONS. In this problem we show that every non-zero finitely generated module \( M \) over a Noetherian ring \( R \) admits a filtration
    \[ 0 = M_0 \subset M_1 \subset M_2 \subset \ldots M_{n-1} \subset M_n = M \]
    such that each subquotient \( M_i/M_{i-1} \cong R/P_i \) for some \( P_i \in \text{Spec} \) \( R \).
    (a) Use Noetherian Induction to reduce to the case that every quotient of \( M \) has a prime cyclic filtration. \([\text{Hint: Recall Noetherian Induction—if we have a counterexample } M, \text{ mod out}} \)
        by a submodule \( N \) maximal with respect the property that \( M/N \) is also a counterexample.\]
    (b) Use (8) to find \( x \in M \) such that \( R/P \cong xR \subset M \) for some \( P \in \text{Spec} \) \( R \).
    (c) Prove that every finitely generated module over a Noetherian ring has a prime cyclic filtration. \([\text{Hint: Splice together } R/P \text{ and a filtration for } M/xR.] \)

(11) PROOF OF THEOREM 1. Fix an arbitrary ring \( R \).
   (a) Prove that if \( P \) is prime, then \( \text{Ass}(R/P) = \{P\} \).
   (b) Show that if \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) is an exact sequence of \( R \)-modules, then
       \( \text{Ass}(M_2) \subset \text{Ass}(M_1) \cup \text{Ass}(M_3) \). \([\text{Hint: If } P = \text{ann} \ x, \text{ consider two cases: either } Rx \cap M_1 = 0 \text{ or if not. Use (a) for the second case.}] \)
   (c) Suppose that \( M_0 \subset M_1 \subset M_2 \subset \ldots M_{n-1} \subset M_n = M \). Show that \( \text{Ass}(M) \subset \bigcup_{i=1}^{n} M_i/M_{i-1} \). \([\text{Hint: Use induction on } n \text{ and (b).}] \)
   (d) Prove that \( \text{Ass}(M) \subset \{P_1, P_2, \ldots, P_n\} \), the prime ideals appearing in a prime cyclic filtration of \( M \).
   (e) Prove the Theorem on the finiteness of \( \text{Ass}(M) \) for Noetherian \( M \) over Noetherian \( R \).

(12) Let \( M \) and \( N \) be two finitely generated modules over a ring \( R \).
   (a) Assume \((R, m)\) is local. Show that if \( M \) and \( N \) are non-zero, then so is \( M \otimes_R N \).
       \([\text{Hint: Consider } M \otimes_R N \to M/mM \otimes_R N/mN \cong M \otimes_R N \cong M/mM \otimes_{R/m} N/mN.] \)
   (b) Show that \((M \otimes_R N)_P \cong M_P \otimes_{R_P} N_P \).
   (c) Show that \( \text{Supp}(M \otimes_R N) = \text{Supp}(M) \cap \text{Supp}(N) \) as subsets of \( \text{Spec} \) \( R \).