Math 512: The Automorphism Group of the Quaternion Group

Professor Karen E. Smith

Recall first that the quaternion group is the smallest subset of the quaternions containing i, j, and k and closed under multiplication. It given by the group presentation: $Q := \{-1, i, j, k \,|\, i^2 = j^2 = k^2 = -1,\ (-1)^2 = 1,\ ij = k\}$. You have already completed the multiplication table for Q. Our next goal is to compute the **Automorphism Group of Q:** the group of all bijections from Q to itself which preserve the group structure.

Philosophical note: You probably already have a feeling that the quaternion group is highly symmetric: the roles of i, j and k are more or less (but not quite) interchangable as far as the group structure is concerned. Intuitively, this says that Q has a large-ish and/or complicated automorphism group. Automorphisms of a group are really just "symmetries" of a group: bijections which preserve the "group-y-ness." In general, the word "automorphism" is usually used to denote a self-bijection of some object which preserves some *algebraic structure* where as the word "symmetry" is usually used to denote a self-bijection which preserves some *geometric structure.* Thinking abstractly, there is no real difference: both automorphisms and symmetries are just some bijections which preserve some kind of structure. So we can think of the "automorphisms of a group" as the "symmetries" of the group. In this exercise, we will try to visualize the group structure of Q geometrically, so that automorphisms of Q really become just symmetries of some geometric object.

1.) Decorate[1] the six sides of cube as follows: pick one vertex v of the cube, and write the letters i, j, k on those faces of the cube meeting v in such a way that as we look at the vertex, the letters i, j, k are alphabetically in counterclockwise order around v (sometimes called "right-hand orientation"). Then write $-i$, $-j$, and $-k$ on the faces opposite i, j, k respectively. Now all six sides of the cube are labelled by one and only one element of Q. Only the elements 1 and -1 are not represented on a face of the cube.

   Show that for every pair $\{x, y\}$ of distinct elements in $Q \setminus \{\pm 1\}$ (with the exception of the case where $y = -x$), there is a unique vertex of the cube contained in the faces labelled by x and y and such that traveling around the vertex from x to y is counterclockwise.

2.) Show that multiplication in Q (other than by ± 1) is represented by the cube as follows. First, if x and y don't have a unique vertex as in (1), then either $x = y$ in which case the product is -1 or $x = -y$ in which case the product is 1. Otherwise, to find the product of x and y, we find the unique right-handed vertex of the cube where two of the three faces meeting at that vertex are labelled x and y (where "right-handed" means that traveling from x to y goes counterclockwise around that vertex.) The product xy will be the third face at that vertex. For example, using our original vertex v, we see that $ij = k$, since i, j, k run counterclockwise around v. Likewise, $jk = i$ since j, k, i run counterclockwise around v and so forth.

[1] Do it with an actual cube!

3.) Show that if we look at a vertex where x, y are oriented clockwise ("left-handed"), then the product xy is $-z$, where z is the third face.

4.) Show that if $f : Q \to Q$ is an automorphism, then f fixes both 1 and -1. Show that an automorphism must take an order four element to an order four element. Can it completely arbitrarily permute the six order four elements?

5). Show that an automorphism of Q is uniquely determined by where it sends i and j.

6.) Let F be a rotational symmetry of the cube. In particular F permutes the faces of the cube (though not completely arbitrarily), and hence their labels $\pm i, \pm j, \pm k$. Show that F determines an unique automorphism of Q in which each $x \in Q \setminus \{\pm 1\}$ is sent to the label on the image face under F. Do all automorphisms of Q arise this way?

7.) Prove that the automorphism group of Q is isomorphic to the (rotational) symmetry group of the cube.

8.) Prove that Aut $Q \cong S_4$. (For 296 students, this is familiar. Here's a hint for everyone else: a rotational symmetry of the cube is uniquely determined by where it sends each of the four "main diagonals" of the cube.)