Math 217: Computing Eigenvalues and their algebraic and geometric multiplicities.
Professor Karen Smith

Definition. Let $V \xrightarrow{T} V$ be a linear transformation. An **eigenvector** of T is a *non-zero* vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ. The scalar λ is the **eigenvalue** of the eigenvector $\vec{v}$.

Definition: The **characteristic polynomial**[1] of an $n \times n$ matrix A is the polynomial

$$\chi_A(x) = \det(A - xI_n).$$

Theorem: Let $V \xrightarrow{T} V$ be a linear transformation of a finite dimensional vector space. A scalar λ is an eigenvalue of T if and only if λ is a root of the characteristic polynomial χ_T.

A. What, here in the Theorem, is meant by the *characteristic polynomial* of T? How is this different from the *characteristic polynomial* of a matrix?

B. For each transformation below, find the characteristic polynomial and the eigenvalues of T.

1. T is left multiplication by $A = \begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix}$.

   Solution note: The char poly is $(2-x)(-6-x)+7 = x^2+4x-5 = (x-1)(x+5)$. So the eigenvalues are 1 and -5.

2. $B = \begin{bmatrix} 2 & 7 & 6 \\ 0 & -1 & -6 \\ 0 & 2 & 7 \end{bmatrix}$.

   Solution note: The char poly is $(2-x)[(-1-x)(7-x)+12 = -(x-2)[x^2-6x+5] = -(x-2)(x-1)(x-5)$. So the eigenvalues are $1, 2$ and 5.

3. The map $\mathbb{R}^3 \to \mathbb{R}^3$ scaling by 3.

   Solution note: The char poly is $(3-x)^3$. The only eigenvalue is 3.

4. For $S = \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix}$, the map $\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ sending A to $S^{-1}AS$.

[1] Mathematicians usually define the characteristic polynomial to be $\det(xI_n - A)$. Note this this polynomial produces $(-1)^n\chi_A(x)$, so the roots are the same. I'll stick with the book's convention even though I find it slightly uglier than the mathematician's. But correct me when I inevitably fail.

Solution note: For this, we need to pick a basis $\mathcal{B}$ and find the $\mathcal{B}$-basis. Say we pick $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. We then compute $T(E_{11}) = \begin{bmatrix} -1 & 7 \\ 0 & 0 \end{bmatrix}, T(E_{12}) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -E_{12}, T(E_{21}) = \begin{bmatrix} -7 & 49 \\ -1 & 7 \end{bmatrix} T(E_{22}) = \begin{bmatrix} 0 & -7 \\ 0 & 1 \end{bmatrix}$. We then arrange this into the $\mathcal{B}$-matrix:

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -7 & 0 \\ 7 & -1 & 49 & -7 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 7 & 1 \end{bmatrix}.$$

We compute its char poly by computing $\det[T]_{\mathcal{B}} - xI_4$ by Laplace expansion along the second column. We get

$$(-1-x)(1-x)(-1-x)(x-1) = (x-1)^2(x+1)^2.$$

This shows that the eigenvalues are $1, 1, -1, -1$ (that is, ± 1 each with algebraic multiplicity 2.

Note that we could have taken any basis $\mathcal{A}$. If we are clever, we might notice that I_2 and S are both eigenvectors (with eigenvalue 1), so they are both excellent choices for (part of) a basis. Also, E_{12} is an eigenvector of eigenvalue -1. These three at least will be very easy to find coordinates for in $\mathcal{A}$, and the $\mathcal{A}$-matrix will have lots of zero. To complete the argument this way, we still need to find a fourth matrix not in the span of I_2, S and E_{12} and find the $\mathcal{A}$ coordinates of its image under T. Try it and see if you get the same characteristic polynomial! (of course you have to, since the char poly doesn't depend on the choice of the basis.

5. $P = \begin{bmatrix} 2 & 1 & \pi & 9 \\ 0 & 2 & 9/7 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

Solution note: Determinants of triangular matrices are easy! The char poly of P is $(2-x)^2(-1-x)^2$ which is the same as $(x-2)^2(x+1)^2$, so the eigenvalues are $2, 2, -1, -1$, or in other words, 2 and -1, both with multiplicity two.

6. $Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 \\ 6 & 5.7 & -1 & 0 \\ 5 & 8 & -3 & -1 \end{bmatrix}$.

Solution note: Determinants of triangular matrices are easy! The char poly of P is $(2-x)^2(-1-x)^2$ which is the same as $(x-2)^2(x+1)^2$, so the eigenvalues are $2, 2, -1, -1$, or in other words, 2 and -1, both with multiplicity two.

7. Differentiation on $\mathcal{P}_4$.

Solution note: Again, we have to represent this linear transformation by a matrix in order to start computing eigenvalues, etc. The standard basis is easiest: we get an triangular matrix with all zeros on the diagonal. So the char poly is $(-x)^5$ and the only eigenvalue is zero, with algebraic multiplicity 5.

8. Projection onto the plane V perpendicular to $\vec{v} = [1\ 1\ 1]^T$.

Solution note: Let $\vec{w}_1$ and $\vec{w}_2$ be linearly independent vectors in V. Then $\{\vec{w}_1, \vec{w}_2, \vec{v}\}$ is a basis and in this basis, the matrix of the projection is diagonal with entries $1, 1, -1$. These are the eigenvalues, and the char poly is $x^2(x-1)$.

C. Define the λ-eigenspace of T. Define algebraic and geometric multiplicity of eigenvalues. What is the relationship between almu and gemu? For each map in B, find both.

Solution note: The algebraic multiplicity of eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial. The geometric multiplicity of eigenvalue λ is the dimension of the λ-eigenspace—equivalently, the dimension of $\ker(A - \lambda I_n)$.

In general, $\text{almu}(\lambda) \geq \text{gemu}(\lambda)$ for every eigenvalue λ.

We list the geometric multiplicities of all the maps in A (we already listed the algebraic multiplicities as we went along). The point is to compute the dimensional of $\ker(A - \lambda I_n)$ which is easy with rank-nullity.

For (1) and (2), all the gemus are 1. This is because all the almus are 1, and gemu is at most almu (and can't be zero, since we are dealing with actual eigenvalues, which *must* have an eigenvector. For (3), clearly scaling by 3 takes every vector to 3 times itself, so the 3-eigenspace is the whole space $\mathbb{R}^3$ and the geometric multiplicity is 3. For (4), the gemu of the eigenvalue 1 is 2, since I_2 and S are linearly independent 1-eigenvectors (and we know gemu(1) can't be more than 2 since almu(1) = 2). For the eigenvalue -1, we use the technique of looking at $A - \lambda I_4$. In this case, we get

$$[T]_{\mathcal{B}} - (-1)I_4 = \begin{bmatrix} 2 & 0 & -7 & 0 \\ 7 & 0 & 49 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 2 \end{bmatrix},$$

which has rank 3. By rank-nullity, the kernel has dimension 1, so the geometric multiplicity of -1 is 1.

For (5), the easiest way is to compute the rank of $P - \lambda I_4$ for each $\lambda = 1, -2$, and use rank-nullity to get the dimension of the kernel (which is the geometric multiplicity). For both eigenvalues, we get the geometric multiplicity is 2. The same thing happens for Q in (6). This tells us that P and Q are both similar to a diagonal matrix with entries $2, 2, -1, -1$, hence similar to each other.

For (7): the gemu of the eigenvalue 0 is the dimension of the kernel, which is 1, since only constant polynomials are in it. Thus differentiation is not diagonalizable.

Solutions to the remaining problems will appear on a future worksheet.

D. Theorem: Eigenvectors of *distinct* eigenvalues are linearly independent. That is, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are eigenvectors with *different* eigenvalues, then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent.

1. Prove the **Corollary:** If A is an $n \times n$ matrix with n different eigenvalues, then A is similar to a diagonal matrix.

2. Prove the **Corollary:** If T is a linear transformation of an n-dimensional space with n different eigenvalues, then T has an eigenbasis.

3. Find an example showing the converse of both Corollaries is false.

4. Prove the theorem in the case of two eigenvectors. Do you see how to generalize your argument to any number of vectors?

5. Prove the following **Theorem:** *Let $V \xrightarrow{T} V$ be a linear transformation of an n-dimensional vector space. If the sum of the geometric multiplicities is n, then T has an eigenbasis, or equivalently, T is diagonalizable.*

6. Which of the maps in B is diagonalizable?

E. Theorem: *Let $\chi_A(x) = (-x)^n + a_1(-x)^{n-1} + \cdots + a_n$ be the characteristic polynomial of an $n \times n$ matrix A. Then $a_1 =$ trace A and $a_n = \det A$.*

1. What does the theorem say in the 2×2 case? State it more cleanly.

2. Prove the theorem in the 2×2 case by direct computation of the characteristic polynomial.

3. **Corollary:** *Suppose that an $n \times n$ matrix A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (possibly repeated if algebraic multiplicity > 1). Then $\det A$ is the product of the eigenvalues and trace A is the sum of the eigenvectors.*
   Prove this. [Hint: factor the characteristic polynomial completely into linear factors.]

4. Prove the 3×3 case of the theorem. Can you prove the $n \times n$ case by induction?