Math 217: Coordinates and $\mathcal{B}$-matrices
Section 5 Professor Karen Smith

Definition. Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. The $\mathcal{B}$-matrix of T is the $n \times n$ matrix

$$B = [T(\vec{v}_1)]_{\mathcal{B}} \quad T(\vec{v}_2)]_{\mathcal{B}} \quad \ldots \quad T(\vec{v}_n)]_{\mathcal{B}}].$$

Computing the $\mathcal{B}$-matrix: Compute one column at a time. For the j-th column:

1. Find $T(\vec{v}_j)$.
2. Express $T(\vec{v}_j)$ as a linear combination of the basis elements $\{\vec{v}_1, \ldots, \vec{v}_n\}$.
3. Write these $\mathcal{B}$-coordinates of $T(\vec{v}_j)$ as a column vector: this is the j-th column of the $\mathcal{B}$-matrix. Do not forget this last step of converting back into $\mathcal{B}$-coordinates! It is easy to forget since the vectors $T(\vec{v}_j) \in \mathbb{R}^n$ are columns already in standard coordinates.

For any vector $\vec{v} \in \mathbb{R}^n$, we can understand T entirely in $\mathcal{B}$-coordinates as follows:

$$[T(\vec{v})]_{\mathcal{B}} = B \cdot [\vec{v}]_{\mathcal{B}}$$

where B is the $\mathcal{B}$-matrix of T.

Theorem. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation (so T is multiplication by some matrix A). Then the $\mathcal{B}$-matrix and the standard matrix A of T are **similar:**

$$B = S^{-1}AS$$

where $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \ldots & \vec{v}_n \end{bmatrix}$. The matrix S is the *change of basis* matrix from $\mathcal{B}$ to the standard basis. The columns of S are the basis vectors of $\mathcal{B}$ expressed in the standard basis.

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A. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 3x + 4y \\ 4x - 3y \end{bmatrix}$.

1. Prove that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} = \{-\vec{e}_1 + 2\vec{e}_2, 2\vec{e}_1 + \vec{e}_2\}$ is a basis for $\mathbb{R}^2$.

   Solution note: The square matrix $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ has rank two, hence its columns are a basis.

2. Compute $\begin{bmatrix} -2 \\ 4 \end{bmatrix}_{\mathcal{B}}$ and $[\vec{e}_1 + 3\vec{e}_2]_{\mathcal{B}}$ and $\left[\frac{7}{13}\vec{v}_1 + \pi\vec{v}_2\right]_{\mathcal{B}}$.

   Solution note: In order, these are $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 17/3 \\ \pi \end{bmatrix}$

3. Find a matrix S which "changes $\mathcal{B}$-coordinates to standard coordinates." That is, S should satisfy

$$S \cdot [\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\{\vec{e}_1.\vec{e}_2\}}$$

   for all vectors $\vec{v} \in \mathbb{R}^2$. Verify explicitly that your S transforms the $\mathcal{B}$-coordinates you found in (3) back into standard coordinates.

Solution note: $S = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is the change of basis matrix form $\mathcal{B}$ to the standard basis. We can verify:

$$\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 17/3 \\ \pi \end{bmatrix} = \begin{bmatrix} -17/3 + 2\pi \\ 34/3 + \pi \end{bmatrix},$$

which are the standard coordinates of $\frac{17}{3}\vec{v}_1 + \pi\vec{v}_2$.

4. Find a matrix P which does the opposite—P should transform standard coordinates into $\mathcal{B}$ coordinates. That is, P should satisfy

$$P \cdot [\vec{v}]_{\{\vec{e}_1 . \vec{e}_2\}} = [\vec{v}]_{\mathcal{B}}$$

for all vectors $\vec{v} \in \mathbb{R}^2$. Check that it works for the vectors in (2). What is the relationship between S and P?

Solution note: P does the reverse of S, so it is given by the inverse matrix: $P = S^{-1} = \frac{-1}{5} \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$.

5. Find the $\mathcal{B}$-matrix of T. Use to compute the image of $\vec{v}_1 + \vec{v}_2$.

Solution note: $[T]_{\mathcal{B}} = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$. In $\mathcal{B}$-coordinates, $\vec{v}_1 + \vec{v}_2$ is written $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So its image in $\mathcal{B}$-coordinates is $\begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$. This is the vector $-5\vec{v}_1 + 5\vec{v}_2$.

6. Find the standard matrix of T. Call it A.

Solution note: $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

7. What is the relationship between $[T]_{\mathcal{B}}$ and the standard matrix for T. Use this to compute $[T]_{\mathcal{B}}$. Do you get the same answer? Alternatively, how could we compute the standard matrix from the $\mathcal{B}$-matrix?

Solution note: $[T]_{\mathcal{B}} = S^{-1}AS$ by the Theorem (memorize this carefully!). So we can compute

$$[T]_{\mathcal{B}} = \frac{-1}{5} \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Doing the multiplication we get the same answer. Alternatively, we can rearrange the statement in the theorem by multiplication on the left by S and on the right by S^{-1}. Then we see

$$A = S[T]_{\mathcal{B}}S^{-1},$$

so we can compute A by multiplying these three matrices together. Of course, we get the same answer, but this is a lot more tedious then finding its columns by seeing where $\vec{e}_1$ and $\vec{e}_2$ go under T.

B. Let $\mathfrak{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3 + \vec{e}_2, \vec{e}_4 + \vec{e}_1\} \subset \mathbb{R}^4$.

1. Prove $\mathfrak{B}$ is a basis for $\mathbb{R}^4$.

Solution note: There are four vectors, so by Theorem 3.3.4, they are a basis for a 4-dimensional space if they span it. But their span includes all the standard vectors, since $\vec{e}_1, \vec{e}_2, \vec{e}_3 = (\vec{e}_3 + \vec{e}_2) - \vec{e}_2$ and $\vec{e}_4 = (\vec{e}_4 + \vec{e}_1) - \vec{e}_1$ are in $\mathfrak{B}$.

2. Write $\vec{x} = [1\ 1\ 1\ 1]^T$ as a linear combination of the elements in $\mathfrak{B}$. What is $[x]_{\mathfrak{B}}$?

Solution note: $\vec{x} = 1(\vec{e}_3 + \vec{e}_2) + 1(\vec{e}_4 + \vec{e}_1)$. So $[x]_{\mathfrak{B}} = [0\ 0\ 1\ 1]^T$.

3. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose matrix with respect the basis $\mathcal{B}$ is $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$. Find the matrix of T in the standard basis (call it A).

Solution note: The columns of the standard matrix will be the $T(\vec{e}_i)$ (expressed in the standard basis). We know $T(\vec{e}_1) = 3\vec{e}_1$, so the first column is $\begin{bmatrix} 3 & 0 & 0 & 0 \end{bmatrix}^T$. Likewise $T(\vec{e}_2) = -\vec{e}_2$, so the second column is $\begin{bmatrix} 0 & -1 & 0 & 0 \end{bmatrix}^T$. The third column will be $T(\vec{e}_3)$ which is a tiny bit harder: we know $\vec{e}_3 = (\vec{e}_3 + \vec{e}_2) - \vec{e}_2$, so $T(\vec{e}_3) = T(\vec{e}_3 + \vec{e}_2) - T(\vec{e}_2) = a(\vec{e}_3 + \vec{e}_2) - (-\vec{e}_2) = (a+1)\vec{e}_2 + a\vec{e}_3$. Note that in computing this, we used the $\mathfrak{B}$-matrix B to compute $T(\vec{e}_3 + \vec{e}_2)$. Thus the third column is $\begin{bmatrix} 0 & a+1 & a & 0 \end{bmatrix}$. Similarly, $T(\vec{e}_4) = T(\vec{e}_4 + \vec{e}_1) - T(\vec{e}_1) = b(\vec{e}_4 + \vec{e}_1) - 3(\vec{e}_1)$, giving us $\begin{bmatrix} b-3 & 0 & 0 & b \end{bmatrix}^T$ for the fourth column. So $A = \begin{bmatrix} 3 & 0 & 0 & b-3 \\ 0 & -1 & a+1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$.

C. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto the plane Λ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

1. Is the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ a basis for Λ? Explain.

2. Can the set in (1) be extended to a basis for all of $\mathbb{R}^3$? Explain.

3. Find a vector $\vec{v} \in \mathbb{R}^3$ perpendicular to Λ.

4. Is the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \bigcup \{\vec{v}\}$ a basis for $\mathbb{R}^3$? Explain.

5. Find the $\mathcal{B}$-matrix of T.

6. Express the matrix for T in standard coordinates in terms of matrices you have already computed here.

Solution note: (1) Yes, the plane has dimension two, and these are two vectors that are linearly independent, since neither is a scalar multiple of the other.

(2) Yes, pick any vector not in the span of these. Then we have three vectors linearly independent in the $3d$ space $\mathbb{R}^3$.

(3) We need a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which has zero dot product with both given vectors. This amounts to solving the linear system

$$x + y + z = 0 \qquad 2x - y + z = 0.$$

Since the coefficient matrix is rank 2, by the rank-nullity theorem, this system has a one-dimensional kernel, so any non-zero vector will span it. (You can also solve it directly). By inspection, we see $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ is a solution, hence a vector perpendicular to the plane.

(4) Yes, these are a basis, since $\vec{v}$ is not in the span of the previous vectors.

(5) $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(6) The standard matrix is $A = S \cdot [T]_{\mathcal{B}} \cdot S^{-1}$.

D. Let V be the subspace of $\mathbb{R}^3$ given by $x_1 + x_2 - 2x_3 = 0$.

1. Find a basis of V in which the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Find the coordinates of $\vec{v}_1$ in the basis $(\vec{v}_2, \vec{v}_3)$, of $\vec{v}_2$ in the basis $(\vec{v}_1, \vec{v}_3)$ and of $\vec{v}_3$ in the basis $(\vec{v}_1, \vec{v}_2)$.