Math 217: Coordinates
Section 5   Professor Karen Smith

A. Important theorem: If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_d\}$ is a basis for W, then every element of W can be written in ONE AND ONLY ONE way as a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_d$.

Let $\mathfrak{B}$ be a basis for a vector space W. Define the $\mathfrak{B}$-coordinates of a vector $\vec{v} \in W$.

> *Solution note:* Read this in the Definitions and Important Theorems Document: Definition right at the start of Section 3.4 (currently Theorem 3.33 on page 20 but since I am editing that document always, page and theorem numbers might change slightly).

B. Below are the some vector spaces you considered last time, and a basis $\mathfrak{B}$ you found for each.

1. The vector space $\mathbb{R}^4$ has basis $\mathfrak{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 + \vec{e}_1\}$.
   (a) Write $\vec{x} = [1\ 1\ 1\ 1]^T$ as a linear combination of the elements in $\mathfrak{B}$. Is this unique? Find the $\mathfrak{B}$-coordinates of $\vec{x}$. What is $[\vec{x}]_{\mathfrak{B}}$?
   (b) Same for $\vec{y} = [1\ 0\ 0\ 1]^T$.

   > *Solution note:* The $\mathfrak{B}$-coordinates of $\vec{x}$ are the column vector $[\vec{x}]_{\mathfrak{B}} = [0\ \ 1\ \ 1\ \ 1]^T$. The $\mathfrak{B}$-coordinates of $\vec{y}$ are the column vector $[\vec{y}]_{\mathfrak{B}} = [0\ \ 0\ \ 0\ \ 1]^T$.

2. The vector space $W = \{\vec{x} \in \mathbb{R}^3 \,|\, \vec{x} \cdot \vec{u} = 0\}$ where $\vec{u} = [1\ \ -1\ \ 0]^T$ has basis $\mathfrak{B} = \{[0\ 0\ 1]^T,\ [1\ 1\ \ 1]^T\}$.
   (a) Express $\vec{x} = [5\ 5\ \ 0]^T$ in this basis. Find $[x]_{\mathfrak{B}}$.

   > *Solution note:* This is the same question rephrased slightly–the answer is $[x]_{\mathfrak{B}} = [5\ \ -5]^T$.

   (b) What are the coordinates of $\vec{x} = [5\ 5\ \ 0]^T$ in the basis $\mathfrak{A} = \{[0\ 0\ 2]^T,\ [1\ 1\ 0]^T\}$?

   > *Solution note:* This is the same question rephrased slightly–the answer is $[x]_{\mathfrak{A}} = [0\ \ 5]^T$.

3. Let V be the vector space of polynomials of degree at most 4 that satisfy $f(0) = 0$. Let $\mathcal{B} = \{x, x^2, x^3, x^4\}$.
   (a) Find the coordinate columns $[3x^3 + x]_{\mathcal{B}}$ and $[x^2(x-1)(x+1)]_{\mathcal{B}}$.

   > *Solution note:* $[3x^3 + x]_{\mathcal{B}} = [1\ \ 0\ \ 3\ \ 0]^T$. Also $[x^2(x-1)(x+1)]_{\mathcal{B}} = [0\ \ -1\ \ 0\ \ 1]^T$.

   (b) Prove that $\{x, x+x^2, x^3, x^4 - x^2\}$ is also a basis for V. Find the coordinates of $x^2(x^2-1)$ in this basis.

> *Solution note:* To see $\{x, x+x^2, x^3, x^4-x^2\}$ is a basis, we note that there are four vectors in this set and we already know the vector space in question has dimension 4. So we only need to show EITHER they span OR they are linearly independent. In class we showed they are linearly independent. Instead now, we show they span: note that the span obviously contains x; the span contains $x^2 = (x+x^2) - x$; the span contains x^3; and the span contains $x^4 = (x^4 - x^2) + x^2$. So the span contains x, x^2, x^3, and x^4, so these four vectors span the whole vector space. The coordinates of $x^2(x^2-1)$ in this basis are $[0\ \ 0\ \ 0\ \ 1]^T$.

4. Let W be the image of $\begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 5 & 4 & 9 & 13 & 6 \\ 7 & 8 & 15 & 23 & 6 \end{bmatrix}$. This has basis $\mathfrak{B} = \{\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}\}$. (Why?). Write $\vec{x} = \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ in this basis. Find $\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}_{\mathfrak{B}}$, $\begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}_{\mathfrak{B}}$ and $\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}_{\mathfrak{B}}$.

> *Solution note:* The matrix has rank 2, which we see by starting to row-reduce and observing (without wasting time going all the way to the rref) that the rref will have two leading ones. This means that the dimension of the image W is two (This is an important theorem; see the book Theorem 3.3.6. Know this one!). So any two linearly independent vectors in W will be a basis. The columns of the matrix span W (another important book theorem from the same section), so we can pick any two linearly independent columns. The first two work. In this basis, we have
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> $$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad [\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
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> Note: in this question we asked the same thing in multiple ways.

5. Is $\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\}$ a basis for $\mathbb{R}^{2\times 2}$? If so, write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in this basis. What are the coordinates of each of the basis elements?

> *Solution note:* Yes. They are obviously linearly independent. Also, they span, since an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the linear combination $aE_{11} + bE_{12} + cE_{21} + dE_{22}$. This shows the cooridinate for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the column vector $[a\ b\ c\ d]^T$. The coordinate column of E_{11} is $\vec{e}_1$, for E_{12} is $\vec{e}_2$, for E_{21} is $\vec{e}_3$, and for E_{22} is $\vec{e}_4$.

6. If $\mathfrak{B}$ is a basis for a vector space W of dimension d, explain how we can use coordinates to think of W as "essentially $\mathbb{R}^d$." Note however, that there may be many different ways to identify W with $\mathbb{R}^d$. Why? That is, there is not necessarily a *canonical* way to think of W as $\mathbb{R}^d$.

> *Solution note:* For this, please read the Definitions and Important Theorems Document, pages 20-21, starting at Section 3.4.

Important Idea: A BASIS GIVES A WAY OF THINKING OF VECTORS IN ANY SUBSPACE AS A "COLUMN VECTOR". THIS LETS US IDENTIFY ANY SUBSPACE OF DIMENSION d WITH $\mathbb{R}^d$. IT GIVES US A CONVENIENT WAY TO EXPRESS ELEMENTS IN VECTOR SPACE, INCLUDING SUBSPACE OF $\mathbb{R}^n$, AND TO TREAT THEM "JUST LIKE $\mathbb{R}^d$."

C. Representing Linear Transformations in other Bases. Let $\vec{u}$ be a unit vector in $\mathbb{R}^2$. Let L be the subspace (line) spanned by $\vec{u} = [u_1 \ \ u_2]^T$. Recall that the map "projection onto L" is a linear transformation

$$\pi : \mathbb{R}^2 \to \mathbb{R}^2$$

whose matrix is $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$.

1. Let $\vec{v}$ be another unit vector such that $\vec{u} \cdot \vec{v} = 0$. Explain why $\mathfrak{B} = \{\vec{u}, \vec{v}\}$ is a basis for $\mathbb{R}^2$. Draw a picture.

   Solution note: The vectors are perpendicular, hence not scalar multiples of eachother. This means that they are linearly independent, and since there are 2 of them and $\mathbb{R}^2$ is 2 dimensional, they are a basis.

2. Compute the kernel and the image of π. Find a basis for each.

   Solution note: The image consists of the line spanned by $\vec{u}$. A basis for the kernel is $\{\vec{u}\}$. The kernel consists of the perpendicular line spanned by $\vec{v}$. A basis for the kernel is $\{\vec{v}\}$.

3. Verify the rank-nullity theorem for π.

   Solution note: $\dim(im) + \dim(ker) = 1 + 1 = 2 = \dim(source)$.

4. Write $\pi(\vec{u})$ in the basis $\mathfrak{B}$.

   Solution note: Since $\pi(\vec{u}) = \vec{u}$, the coordinates of $\pi(\vec{u}) = [1 \ \ 0]^T$.

5. Write $\pi(v)$ in the basis $\mathfrak{B}$.

   Solution note: Since $\pi(\vec{v}) = \vec{0}$, the coordinates of $\pi(\vec{v}) = [0 \ \ 0]^T$.

6. Find the $\mathfrak{B}$ matrix of π. Why might expressing π in this basis be more convenient than the standard basis?

   Solution note: $[\pi]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Lots of zeros makes computation fast and easy! Also we easily see that the rank is 1.

7. Explain how to use the $\mathfrak{B}$ matrix of π to compute $\pi(\vec{w})$ where $\vec{w}$ is any vector in $\mathbb{R}^2$.

   Solution note: The technique is: write $\vec{w}$ in $\mathfrak{B}$-coordinates, then multiply by the $\mathfrak{B}$ matrix. The result will be the column of $\mathfrak{B}$-coordinates of $\pi(\vec{w})$. You can rewrite these in standard coordinates if you want.

8. Now if $\mathfrak{S}$ is the standard basis for $\mathbb{R}^2$, find the $\mathfrak{S}$ matrix of π.

> *Solution note:* We did this on a previous worksheet: it was much harder! The answer is $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$.

9. What is the relationship between the $\mathfrak{B}$ matrix and the $\mathfrak{S}$ matrix?

> *Solution note:* They are similar! if we represent a linear transformation in two different bases, the matrices are always similar.

10. Define what it means that two $n \times n$ matrices A and B are **similar** and explain what it has to do with this problem.

> *Solution note:* Matrices A and B are similar means that there exists an invertible matrix S such that $B = S^{-1}AS$. In this case, we have $B = S^{-1}AS$ where $S = [\vec{u}\ \vec{v}]$. This is from a theorem at the end of Section 3.4 in the book.

Important Idea: COORDINATES LET US MODEL ANY n-DIMENSIONAL VECTOR SPACE V ON $\mathbb{R}^n$. Likewise, coordinates let us model any linear transformation $V \xrightarrow{T} V$ as a matrix multiplication. Even in the case $V = \mathbb{R}^n$, a non-standard choice of coordinates gives a different way to model $\mathbb{R}^n$ and its linear transformations which might be more helpful in a particular problem.

Definition: If $\mathfrak{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of V, the $\mathfrak{B}$-matrix is the $n \times n$ matrix whose columns are $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ EXPRESSED IN THE BASIS $\mathfrak{B}$. That is, the $\mathfrak{B}$ matrix of T is

$$\big[[T(\vec{v}_1)]_{\mathfrak{B}} \quad [T(\vec{v}_2)]_{\mathfrak{B}} \quad \ldots \quad [T(\vec{v}_n)]_{\mathfrak{B}}\big].$$

How do we use this matrix to compute $T(\vec{v})$ for any vector $\vec{v} \in V$?

D. Let $\mathcal{P}_4$ be the space of polynomials of degree 4 or less.

1. Prove that the map $D : \mathcal{P}_4 \to \mathcal{P}_4$ sending $f \mapsto f - f'$ is a linear transformation.
2. Find the matrix of D with respect to the basis $\{1, x, x^2, x^3, x^4\}$.
3. Is D is invertible? Explain. [Your matrix in (2) might be useful for this but there are many correct approaches.]
4. Use your answer to (2) to find the matrix of D^{-1} with respect to the given basis.
5. Use your answer to (4) to give a formula for $D^{-1}(a + bx + cx^2 + dx^3 + ex^4)$.

> *Solution note:* This problem got bumped to a future worksheet.