Math 412. Group Actions and Orbits

**Definition:** Let \((G, \circ)\) be any group. Let \(X\) be any set. We say that the group \(G\) acts on \(X\) if there is a map
\[G \times X \to X \quad (g, x) \mapsto g \cdot x,\]
satisfying the following two axioms:

1. \(e \cdot x = x\) for all \(x \in X\); and
2. \(h \cdot (g \cdot x) = (h \circ g) \cdot x\) for all \(g, h \in G\) and all \(x \in X\).

Fix an action of a group \(G\) on a set \(X\). Consider a point \(x \in X\). Consider one point \(x \in X\).

**Definition:** The orbit of \(x\) is the subset of \(X\)
\[O(x) := \{g \cdot x \mid g \in G\} \subset X.\]

A. Let \(D_4\) be the symmetry group of the square. The group \(D_4\) acts on the set \(X\) of points of the square in a canonical way. Note that \(X\) is an infinite set.

1. Draw a picture of the square in the Cartesian plane so its vertices are \(\{(\pm 1, \pm 1)\}\). Explain the canonical action of \(D_4\) on the square.
2. Compute the orbit of each the following types of points (and sketch): the origin, a vertex, a non-zero point on a diagonal of the square, a non-zero point on the horizontal axis of symmetry, a non-zero point not on any axis of symmetry.
3. What is the largest number of points any orbit can have? Find an explicit point whose orbit is achieves this value.
4. True or False: under the given action of \(D_4\) on the square, all vertices have the same orbit.

B. Consider the rotation group \(SO_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \mid \theta \in [0, 2\pi) \right\}\). The group \(SO_2(\mathbb{R})\) acts on the plane \(\mathbb{R}^2\) by matrix multiplication: for \(A \in SO_2(\mathbb{R})\), \(A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}\).

1. Sketch the plane, and the orbits of the points \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{and} \begin{bmatrix} 0 \\ 0 \end{bmatrix}\).
2. Is every point of the plane in some orbit?
3. For two points, \(\begin{bmatrix} a \\ b \end{bmatrix}\) and \(\begin{bmatrix} c \\ d \end{bmatrix}\) in the plane, how can you tell if the orbits \(O(\begin{bmatrix} a \\ b \end{bmatrix}) = O(\begin{bmatrix} c \\ d \end{bmatrix})\)?
4. Can two different orbits of this group action intersect?
5. True or False: The set \(\mathbb{R}^2\) is the disjoint union of its distinct orbits under the given action of \(SO_2(\mathbb{R})\).

C. Let a group \(G\) act on a set \(X\).

1. Prove that the relation “\(x \sim y\) if \(x \in O(y)\)” is an equivalence relation on \(X\).
2. Prove that if \(x \in O(y)\), then \(O(x) = O(y)\).
3. Prove that for any two elements \(x, y \in X\), either their orbits either coincide exactly or are disjoint.
(4) Prove that $X$ is the disjoint union of its distinct orbits.

D. There can be different actions of the same group $G$ on the same set $X$. For example, the group $\mathbb{Z}_2$ can act on the Cartesian plane as follows:

$$\mathbb{Z}_2 \times \mathbb{R}^2 \to \mathbb{R}^2 \quad [1]_2 \cdot (x, y) \mapsto (y, x)$$

(and of course the identity $[0]_2$ does nothing). A different action of $\mathbb{Z}_2$ on $\mathbb{R}^2$ is given by

$$\mathbb{Z}_2 \times \mathbb{R}^2 \to \mathbb{R}^2 \quad [1]_2 \cdot (x, y) \mapsto (-x, -y)$$

(and again, $[0]_2$ does nothing).

(1) Verify that these are both group actions. Describe them geometrically.

(2) Find another group action, different from these, of $\mathbb{Z}_2$ on $\mathbb{R}^2$. There are many possibilities; check on a neighboring group to see what they came up with as well.

(3) For each of the three actions in play here, describe the orbits. How many elements can be in an orbit?

E. Prove that if a group $G$ acts on a set $X$, then for every $x \in X$, the cardinality of the orbit satisfies

$$|O(x)| \leq \min(|G|, |X|),$$

where $|X|$ means the cardinality of $X$. Do all orbits necessarily have the same cardinality?

F. Let $S_4$ be the group of permutations of the set $\{1, 2, 3, 4\}$. There is a canonical action of $S_4$ on the set $X = \{1, 2, 3, 4\}$ defined by $\sigma \cdot x = \sigma(x)$.

(1) Verify that this is a group action.

(2) Find the orbit of the element $4 \in X$. Explain why this group action has only one orbit.

(3) Now let $\mathcal{P}(X)$ be the set of subsets of $\{1, 2, 3, 4\}$. List out the elements of $\mathcal{P}(X)$. What is the cardinality of $\mathcal{P}(X)$? Describe a naturally induced action of $S_4$ on $\mathcal{P}(X)$. Verify that your action satisfies the axioms of an action.

(4) Find the orbit of the set $\{1\} \in \mathcal{P}(X)$ under the action of $S_4$ you described in (3). What is the cardinality of this orbit?

(5) Find the orbit of $\{1, 2\} \in \mathcal{P}(X)$. What is the cardinality of this orbit?

(6) Partition $\mathcal{P}(X)$ up into its orbits for this action. Put differently, express $\mathcal{P}(X)$ as a disjoint union of its distinct orbits.

G. Some More Natural Actions and Orbits.

(1) Let the group of positive real numbers, $\mathbb{R}_{>0}$, act on $\mathbb{R}$ by multiplication. What is the orbit of $1 \in \mathbb{R}$? What is the orbit of $0$? What is the orbit of $-11 \in \mathbb{R}$? How many distinct orbits are there? Describe the decomposition of $\mathbb{R}$ into distinct orbits for this action.

(2) Let the group $(\mathbb{Z}^2, +)$ act on the Cartesian plane $\mathbb{R}^2$ by translations: $(a, b) \cdot (x, y) = (a+x, y+b)$. Describe the orbits.

(3) Let $O_2(\mathbb{R})$ be the group of orthogonal transformations of in $\mathbb{R}^2$, acting on the set of all triangles in the plane. What is the order of a fixed triangle $T$.

H. Consider the canonical action of $S_5$ on the set $X = \{1, 2, 3, 4, 5\}$. Let $G$ be the subgroup of $S_5$ generated by $\{(12), (345)\}$.

(1) Explain why $G$ acts on $X$.

(2) Find the orbit of each element of $X$ under $G$.

(3) Partition $X$ up into its orbits.