

## MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 10 SOLUTIONS

**Problem 1.** (a.) We have that  $Z \subseteq X$  since  $xw - yz \in \langle x, y \rangle$ , and

$$k[w, x, y, z]/\langle x, y \rangle \simeq k[w, z]$$

shows that  $\langle x, y \rangle$  is a prime ideal. Thus,  $\mathbb{I}(Z) = \langle x, y \rangle$  and  $Z \simeq \mathbb{A}^2$  is an irreducible subvariety of  $X$  of dimension two. Since  $X$  has dimension three,  $Z$  defines a prime (Weil) divisor on  $X$ .

(b.) Clearly,  $Z$  does not intersect the affine open sets  $U_x = X \setminus \mathbb{V}(x)$  and  $U_y = X \setminus \mathbb{V}(y)$ . Thus, we may take 1 to be a local equation for  $Z$  on  $U_x$  and  $U_y$ . On  $U_z = X \setminus \mathbb{V}(z)$ , we have

$$y = \frac{w}{z}x \quad \text{mod } \langle xw - yz \rangle.$$

Thus,  $x$  is a local equation for  $Z$  on  $U_z$ . Lastly, on  $U_w = X \setminus \mathbb{V}(w)$ , we have

$$x = \frac{z}{w}y \quad \text{mod } \langle xw - yz \rangle,$$

and  $y$  is a local equation for  $Z$  on  $U_w$ . We conclude  $Z$  is locally principal on  $X \setminus \{(0, 0, 0, 0)\} = U_w \cup U_x \cup U_y \cup U_z$ .

(c.) Let  $\mathfrak{m}$  be the maximal ideal of functions on  $X$  vanishing at  $(0, 0, 0, 0)$ . Consider the Zariski cotangent space  $(T_{(0,0,0,0)}X)^* = \mathfrak{m}/\mathfrak{m}^2$ . Since  $X$  is singular at  $(0, 0, 0, 0)$ , we have that  $\mathfrak{m}/\mathfrak{m}^2$  is a four dimensional vector space with a basis given by  $\bar{w}, \bar{x}, \bar{y}, \bar{z}$ . If  $Z$  were locally principal in a neighborhood of  $(0, 0, 0, 0)$ , the image of  $\mathbb{I}(Z)$  in  $\mathfrak{m}/\mathfrak{m}^2$  would have dimension at most one. However, this image is spanned by the basis vectors  $\bar{x}$  and  $\bar{y}$ , so has dimension two. Thus,  $Z$  cannot be locally principal in a neighborhood of the origin.

(d.) On a smooth variety, every divisor is locally principal. However,  $X$  is not smooth. The singular locus is given by  $\mathbb{V}(x, y, z, w) = \{(0, 0, 0, 0)\}$ , which has codimension three. Since  $X$  is a complete intersection, the quoted theorem implies that  $X$  is normal.

**Problem 2.** Since  $\mathbb{P}^m \times \mathbb{P}^n$  is smooth, the Picard group and the class group are the same. We will calculate the latter. Let  $x_0, \dots, x_m$  be homogeneous coordinates on  $\mathbb{P}^m$ , and  $y_0, \dots, y_n$  homogeneous coordinates on  $\mathbb{P}^n$ . Let  $F = \mathbb{V}(x_0)$  and  $G = \mathbb{V}(y_0)$ , and set  $U = \mathbb{P}^m \times \mathbb{P}^n \setminus (F \cup G) \simeq \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$  which has affine coordinates  $x_1, \dots, x_m, y_1, \dots, y_n$ .

Let  $D$  be a prime (Weil) divisor. Then  $D = \mathbb{V}(f(x_0, \dots, x_m, y_0, \dots, y_n))$  for an irreducible bi-homogeneous polynomial  $f \in k[x_0, \dots, x_m, y_0, \dots, y_n]$ , unique up to multiplication by a nonzero scalar. If  $D$  intersects  $U$ , one can see this by taking the bi-homogenization of a local equation for  $D$  on  $U$ . We define the bi-degree  $(a, b) \in \mathbb{Z}^2$  of  $D$  to be the bi-degree of  $f$ , and we extend the notion of bi-degree to arbitrary divisors linearly.

If  $g \in k[x_0, \dots, x_m, y_0, \dots, y_n]$  is any bi-homogeneous polynomial, and  $g = g_1^{r_1} \cdots g_s^{r_s}$  is a factorization into irreducibles, then each  $g_i$  is also bi-homogeneous. If  $D_i$  is the prime divisor  $\mathbb{V}(g_i)$ , we can define the divisor of  $g$  to be

$$\operatorname{div}(g) = \sum_{i=1}^s r_i D_i.$$

Further, any effective divisor can be written  $\operatorname{div}(g)$  for some bi-homogeneous polynomial  $g$  of the same bi-degree.

Recall that any rational function  $\phi$  on  $\mathbb{P}^m \times \mathbb{P}^n$  can be written as the quotient of bi-homogeneous polynomials of the same bi-degree. Thus, the bi-degree of  $\operatorname{div}(\phi)$  is  $(0, 0) \in \mathbb{Z}^2$ , and there is a well defined homomorphism

$$\Phi : \operatorname{Cl}(\mathbb{P}^m \times \mathbb{P}^n) \rightarrow \mathbb{Z}^2$$

which takes the class of a divisor to its bi-degree. In fact, this is an isomorphism. Suppose  $E$  is a divisor of bi-degree  $(0, 0)$ . We can write  $E = E_1 - E_2$  for effective divisors  $E_1$  and  $E_2$  of the same bi-degree. Write  $E_1 = \operatorname{div}(h_1)$  and  $E_2 = \operatorname{div}(h_2)$  for bi-homogeneous polynomials  $h_1, h_2$  of the same bi-degree. Then  $\frac{h_1}{h_2}$  is a rational function on  $\mathbb{P}^m \times \mathbb{P}^n$  with  $\operatorname{div}(\frac{h_1}{h_2}) = E_1 - E_2 = E$ . Thus,  $\Phi$  is injective. Also, we have that  $F$  has bi-degree  $(1, 0)$  and  $G$  has bi-degree  $(0, 1)$ , so  $\Phi$  is surjective.

In conclusion, we have  $\operatorname{Pic}(\mathbb{P}^m \times \mathbb{P}^n) = \operatorname{Cl}(\mathbb{P}^m \times \mathbb{P}^n) \simeq \mathbb{Z}^2$ . We can take as generators the classes of  $F = (\text{hyperplane in } \mathbb{P}^m) \times \mathbb{P}^n$  and  $G = \mathbb{P}^m \times (\text{hyperplane in } \mathbb{P}^n)$ .

**Problem 3.** (a.) The blow-up  $\tilde{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$  of the affine plane at the origin is covered by two affine patches, which we label  $U_x$  and  $U_y$ . Each patch is itself isomorphic to  $\mathbb{A}^2$ , and we may take  $x, \frac{y}{x}$  as coordinates on  $U_x$ , and  $\frac{x}{y}, y$  as coordinates on  $U_y$ .

Let  $L = \mathbb{V}(ax + by)$ . Then, on  $U_x$  we have  $ax + by = x \cdot (a + b\frac{y}{x})$ . On this affine patch,  $x$  is a local defining equation for  $E$ , and  $a + b\frac{y}{x}$  is a local defining equation for the proper transform  $\tilde{L} = \overline{\pi^{-1}(L \setminus \{(0, 0)\})}$  of  $L$ . Similar calculations hold on  $U_y$ . We conclude that

$$\pi^*L = \tilde{L} + E,$$

so that the coefficient of  $E$  in  $\pi^*L$  is one.

(b.) We will assume  $f_d$  is nonzero. On  $U_x$ , we have

$$\begin{aligned} f(x, y) &= f_d(x, y) + f_{d+1}(x, y) + \cdots + f_n(x, y) \\ &= x^d f_d(1, \frac{y}{x}) + x^{d+1} f_{d+1}(1, \frac{y}{x}) + \cdots + x^n f_n(1, \frac{y}{x}) \\ &= x^d \left( f_d(1, \frac{y}{x}) + x f_{d+1}(1, \frac{y}{x}) + \cdots + x^{n-d} f_n(1, \frac{y}{x}) \right). \end{aligned}$$

On this affine patch,  $x$  is a local defining equation for  $E$ , and

$$f_d(1, \frac{y}{x}) + x f_{d+1}(1, \frac{y}{x}) + \cdots + x^{n-d} f_n(1, \frac{y}{x})$$

is a local defining equation for the proper transform  $\tilde{C} = \overline{\pi^{-1}(C \setminus \{(0,0)\})}$ . Similar calculations hold on  $U_y$ . We conclude that

$$\pi^*C = \tilde{C} + dE.$$

In particular, the coefficient of  $E$  in  $\pi^*C$  is the multiplicity  $d$  of  $C$  at the origin.

(c.) We may assume  $C$  and  $f$  are as in the previous part with  $d = 0$ . Then we see  $\pi^*C = \tilde{C}$ , and the coefficient of  $E$  in  $\pi^*C$  is zero. Note that  $\pi$  induces an isomorphism between  $\tilde{C}$  and  $C$ .

**Problem 4.** (a.) The points of  $\text{Spec}(\mathbb{Z})$  are the prime ideals of  $\mathbb{Z}$ . These either have the form  $\langle p \rangle$  for a prime number  $p$ , or  $\langle 0 \rangle$ . Since the prime ideals of the form  $\langle p \rangle$  are maximal, these points are closed. However, since  $\langle 0 \rangle \subseteq \langle p \rangle$  for any prime  $p$ , the closure of the point  $\langle 0 \rangle$  in  $\text{Spec}(\mathbb{Z})$  is all of  $\text{Spec}(\mathbb{Z})$ .

(b.) We will use problem 7 (d.) from problem set 6. If  $p$  is a prime, then denote by  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  the finite field with  $p$  elements. The fiber over  $\langle p \rangle$  is the affine scheme corresponding to the ring

$$\mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3).$$

The generic fiber is the affine scheme corresponding to the ring

$$\mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[x, y, z]/(x^3 + y^3 + z^3).$$

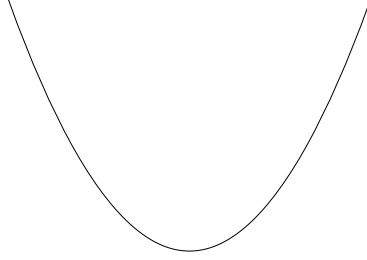
(c.) Note that, only in characteristic 3,  $x^3 + y^3 + z^3$  is not reduced or irreducible. Thus, the affine scheme corresponding to the fiber over  $\langle 3 \rangle$  is not reduced, whereas the ring corresponding to any other fiber is even a domain. The generic fiber is in the “good” set. It is called the generic fiber because its properties are representative of the properties of the fiber over a general prime (i.e. the properties which hold in all but finitely many prime characteristics).

**Problem 5.** (a.) and (b.) In the following, the geometric intuition for separating points is that the image curve does not intersect itself. The geometric intuition for separating tangent vectors is that the image curve has not pinch points or cusps.

- $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  sending  $t \mapsto (t, t^2)$  : This map is clearly injective. Further, at  $p \in \mathbb{A}^1$ , the map on tangent spaces is given by

$$\begin{pmatrix} 1 \\ 2p \end{pmatrix}$$

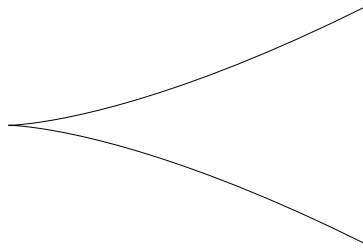
which is also injective for all  $p$ . Thus, this map separates points, and separates tangent vectors everywhere.



- $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  sending  $t \mapsto (t^2, t^3)$  : Again, this map is injective. If  $(t_1^2, t_1^3) = (t_2^2, t_2^3)$ , we have  $t_1^2 = t_2^2$ , so that  $t_1 = \pm t_2$ . Using also that  $t_1^3 = t_2^3$ , we conclude  $t_1 = t_2$ . The map on tangent spaces at  $p \in \mathbb{A}^1$  is given by

$$\begin{pmatrix} 2p \\ 3p^2 \end{pmatrix}$$

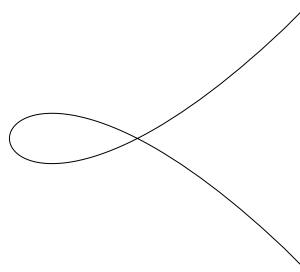
which is injective everywhere except  $p = 0$ . Thus, this map separates points, and separates tangent vectors on  $\mathbb{A}^1 \setminus \{0\}$ .



- $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  sending  $t \mapsto (t^2 - 1, t(t^2 - 1))$  : This map does not separate points since both 1 and  $-1$  map to  $(0, 0)$ . However, the map on tangent spaces at  $p \in \mathbb{A}^1$  is given by

$$\begin{pmatrix} 2p \\ 3p^2 - 1 \end{pmatrix}$$

which is injective everywhere. Thus, this map does not separate points, but it does separate tangent vectors everywhere.



(c.) Let  $\pi : X \rightarrow Y$  be a surjective morphism of projective varieties that separates points and tangent vectors. Then  $\pi$  is a set-theoretic bijection, and hence has finite fibers. Since  $X$  is projective, it follows that  $\pi$  is a finite morphism.

Fix  $p \in X$ , and let  $q = \pi(p)$ . It suffices to show the induced map  $\pi^* : \mathcal{O}_{Y,q} \rightarrow \mathcal{O}_{X,p}$  is an isomorphism, since this implies that  $\pi$  induces an isomorphism between sufficiently small open

neighborhoods of  $p$  and  $q$ . Since  $\pi$  is dominant,  $\pi^*$  is automatically injective. We need to show it is surjective.

Let  $\mathfrak{m}_q$  denote the maximal ideal of  $\mathcal{O}_{Y,q}$ , and  $\mathfrak{m}_p$  the maximal ideal of  $\mathcal{O}_{X,p}$ . Since  $\pi$  separates tangent vectors, the map

$$(d\pi)_p : T_p X \rightarrow T_q Y$$

is injective. Thus, the map

$$\mathfrak{m}_q \longrightarrow \mathfrak{m}_q/\mathfrak{m}_q^2 \xrightarrow{\pi^*} \mathfrak{m}_p/\mathfrak{m}_p^2$$

is surjective. By Nakayama's lemma, we have  $\mathfrak{m}_q \mathcal{O}_{X,p} = \mathfrak{m}_p$ . Since  $\pi$  is a finite morphism,  $\pi^*$  makes  $\mathcal{O}_{X,p}$  into a finitely generated module over  $\mathcal{O}_{Y,q}$ . Thus, since the class of 1 is a generator of  $\mathcal{O}_{X,p}/\mathfrak{m}_q \mathcal{O}_{X,p} = \mathcal{O}_{X,p}/\mathfrak{m}_p = k$ , another application of Nakayama's lemma shows that 1 generates  $\mathcal{O}_{X,p}$  as an  $\mathcal{O}_{Y,q}$  module. In other words,  $\pi^*$  is surjective. Thus,  $\pi$  is an isomorphism.

**Problem 6.** (a.) The prime ideals of  $\mathbb{R}[x]$  have the form  $\langle f \rangle$ , where  $f$  is one of:

- (i)  $f(x) = 0$ ;
- (ii)  $f(x) = x - \lambda$ , where  $\lambda \in \mathbb{R}$ ;
- (iii)  $f(x) = x^2 + bx + c$ , where  $a, b \in \mathbb{R}$  and  $b^2 - 4c < 0$ .

In the first case, the fiber over  $\langle 0 \rangle \subseteq \mathbb{R}[x]$  is  $\langle 0 \rangle \subseteq \mathbb{C}[x]$ , and corresponds algebraically to the affine scheme associated with the ring  $\mathbb{R}(x) \otimes_{\mathbb{R}[x]} \mathbb{C}[x] = \mathbb{C}(x)$ . In case (ii), the fiber over  $\langle x - \lambda \rangle$  corresponds algebraically to the affine scheme associated with the ring

$$\mathbb{R}[x]/\langle x - \lambda \rangle \otimes_{\mathbb{R}[x]} \mathbb{C}[x] = \mathbb{C}[x]/\langle x - \lambda \rangle.$$

Geometrically, this means that the fiber over the "point"  $\langle x - \lambda \rangle$  is the single point  $\lambda \in \mathbb{A}_{\mathbb{C}}^1$ . In case (iii), we can write  $f(x) = x^2 + bx + c = (x - \alpha)(x - \bar{\alpha})$  where  $\alpha \in \mathbb{C} \setminus \mathbb{R}$ . Algebraically, the fiber over  $\langle x^2 + bx + c \rangle$  is the affine scheme corresponding to the ring

$$\mathbb{R}[x]/\langle x^2 + bx + c \rangle \otimes_{\mathbb{R}[x]} \mathbb{C}[x] = \mathbb{C}[x]/\langle (x - \alpha)(x - \bar{\alpha}) \rangle.$$

Geometrically, the fiber over the "point"  $\langle x^2 + bx + c \rangle$  is the two-point set  $\{\alpha, \bar{\alpha}\} \subseteq \mathbb{A}_{\mathbb{C}}^1$ .

(b.) Conjugation of the coefficients of a polynomial over  $\mathbb{C}$  yields a ring homomorphism

$$\begin{aligned} \Phi : \mathbb{C}[x] &\rightarrow \mathbb{C}[x] \\ g(x) &\mapsto \overline{g(x)}. \end{aligned}$$

This is equivalent to a morphism of affine schemes  $\Phi : \text{Spec } \mathbb{C}[x] \rightarrow \text{Spec } \mathbb{C}[x]$ . Since  $\Phi^2 = \text{Id}$ , there is an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $\text{Spec } \mathbb{C}[x]$  where the nontrivial element acts by  $\Phi \in \text{Aut}(\text{Spec } \mathbb{C}[x])$ . This action takes  $\langle 0 \rangle$  to  $\langle 0 \rangle$ , and  $\langle x - \alpha \rangle$  to  $\langle x - \bar{\alpha} \rangle$  for  $\alpha \in \mathbb{C}$ . Thus, the fixed points of this action are  $\langle 0 \rangle$  and  $\langle x - \lambda \rangle$  for  $\lambda \in \mathbb{R}$ . There are three different types of orbits of this action:

- (i)  $\{\langle 0 \rangle\}$ ;
- (ii)  $\{\langle x - \lambda \rangle\}$  for  $\lambda \in \mathbb{R}$ ;

(iii)  $\{\langle x - \alpha \rangle, \langle x - \bar{\alpha} \rangle\}$  for  $\alpha \in \mathbb{C} \setminus \mathbb{R}$ .

We can associate the quotient space (i.e. the space of orbits) with  $\text{Spec } \mathbb{R}[x]$  since there is a bijection between the orbits and the fibers of  $\phi$ . It is easy to see that this association is a homeomorphism.

(c.) One simple generalization is whenever we have a Galois extension of fields  $k \subset L$ , the closed points of  $\text{Spec } k[X]$  are the ideals  $(f)$  generated by irreducible polynomials over  $k$ . The fiber over these correspond to the finite set of points in the affine line over  $L$  consisting of the roots of  $f$ . Thus, the closed points of  $k[x]$  are basically sets of  $L/k$  conjugate elements of  $L$ . The Galois group acts on  $\text{Spec } L[x]$  and the quotient is homeomorphic to  $\text{Spec } k[x]$ . The proof is exactly the same as the case of  $\mathbb{R} \subset \mathbb{C}$ .

Actually, if  $R$  is any normal domain finitely generated over  $k$  and  $L$  is any Galois extension of  $k$ , the induced map  $\text{Spec } L[x] \rightarrow \text{Spec } k[x]$  is also the quotient map by the Galois group in the same way!

For some further reading, you could try Kollár, Smith and Corti, *Rational and Nearly Rational Varieties*, Cambridge University Press, 2004. The statement above is essentially the content of exercise 1.8 there (whose detailed solution is on page 182). More general statements can be found in Section 3.4 of this text and the references cited there.