MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 1 SOLUTIONS

Problem 1. (a.) The $(t+1)\times(t+1)$ minors $m_1(A),\ldots,m_k(A)$ of an $n\times m$ matrix A are polynomials in the entries of A, and $m_i(A)=0$ for all $i=1,\ldots,k$ if and only if $\operatorname{rank}(A)\le t$. In other words, $X_t=\mathbb{V}(m_1,\ldots,m_k)$ is an algebraic subset of k^{nm}.

(b.) SU_2 can be described as the set of complex matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{such that } ad-bc=1,\ c=-\bar{b},\text{ and } d=\bar{a}.$$

Suppose, by way of contradiction, that SU_2 is an algebraic subset of $\mathbb{C}^4$. Then, in particular, the intersection of SU_2 with $\mathbb{V}(b,c)\simeq\mathbb{C}^2$ is also an algebraic set. In other words, the set of pairs $(a,d)\in\mathbb{C}^2$ with $ad=1$ and $d=\bar{a}$ is an algebraic set. Now, using a as a coordinate, we can identify $\mathbb{V}(ad-1)$ in $\mathbb{C}^2$ with the Zariski open subset $\mathbb{C}\setminus\{0\}$ of $\mathbb{C}$. It follows that $Z:=\{\ a\in\mathbb{C}\setminus\{0\}\mid \bar{a}=a^{-1}\ \}\cup\{0\}$ is Zariski closed in $\mathbb{C}$. Since $2\neq 1/2$, we see that $Z\neq\mathbb{C}$. The Zariski closed subsets of $\mathbb{C}$ are either $\mathbb{C}$ or finite, hence Z must be a finite set. But this is absurd, as the unit circle $\mathrm{S}^1=\{\ a\in\mathbb{C}\mid |a|=1\}\subset Z$ is an infinite subset! Thus, we conclude that SU_2 is not an algebraic subset of $\mathbb{C}^4$.

However, SU_2 is an algebraic set of $\mathbb{R}^8$. Let a_1 and a_2 be the real and imaginary parts of a, respectively, and similarly for b,c,d. Taking the real and imaginary parts of the equations above, we see that the following polynomials in $\mathbb{R}[a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2]$ cut out SU_2:

$$a_1d_1-b_1c_1-a_2d_2+b_2c_2=1 \quad;\quad a_2d_1+a_1d_2-b_1c_2-b_2c_1=0$$

$$c_1=-b_1 \quad;\quad c_2=b_2 \quad;\quad d_1=a_1 \quad;\quad d_2=-a_2.$$

Problem 2. (a.) Let $I=\langle x^3-z,x^2-y\rangle$. It is clear that $V\subset\mathbb{V}(I)$; alternatively, if $(x,y,z)\in\mathbb{V}(I)$, then $(x,y,z)=(x,x^2,x^3)$ is the image of $x\in\mathbb{A}^1$.

(b.) To see that $\mathbb{I}(V)=\mathbb{I}(\mathbb{V}(I))=I$, it suffices to show I is prime (and hence radical). Consider the map $\Phi:k[x,y,z]\to k[t]$ given by $f(x,y,z)\mapsto f(t,t^2,t^3)$. This map is surjective (t is the image of x), and it is enough to check that I is the kernel.

Let $f(x,y,z)\in k[x,y,z]$. Then we can consider f as an element in $R[z]$, where $R=k[x,y]$. Using the division algorithm, we can write

$$f(z)=(x^3-z)g(z)+r(z),$$

where $\deg_z r < \deg_z(x^3 - z) = 1$. So $\deg_z r = 0$, and we see that $r \in R$ is a polynomial in x and y only. Similarly we can write $r(y) = (x^2 - y)h(y) + s(y)$, where $s \in k[x]$. So we get

$$f(x, y, z) = (x^3 - z)g(x, y, z) + (x^2 - y)h(x, y) + s(x).$$

Now, if $\Phi(f) = s(t) = 0$ if and only if $s(x) = 0$, i.e., $f \in I$. Thus, it follows that Φ induces an isomorphism of $k[x, y, z]/I$ onto the integral domain $k[t]$, and hence I is a prime ideal.

Problem 3. (a.) The nonempty open subsets of $\mathbb{A}^1$ in the Zariski topology of have the form $\mathbb{A}^1 \setminus \{p_1, \ldots, p_n\}$. A basis for the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ is given by sets of the form

$$\mathbb{A}^1 \setminus \{p_1, \ldots, p_n\} \times \mathbb{A}^1 \setminus \{q_1, \ldots, q_m\}.$$

Consider the Zariski open set $\mathbb{A}^2 \setminus \Delta$, where $\Delta = V(x - y)$. Since k is algebraically closed, it is infinite (this is crucial); thus, every basis element intersects Δ. In particular, $\mathbb{A}^2 \setminus \Delta$ cannot be open in the product topology – it does not contain any basis elements at all!

(b.) Denote the coordinates of $\mathbb{A}^n$ by $x_1, \ldots, x_n$, and the coordinates of $\mathbb{A}^m$ by $x_{n+1}, \ldots, x_{n+m}$. Let $V \subset \mathbb{A}^n$ be given by $I = \langle f_1, \ldots, f_l \rangle$ and $W \subset \mathbb{A}^m$ by $J = \langle g_1, \ldots, g_k \rangle$. Then the subset $V \times W$ of $\mathbb{A}^{n+m}$ is determined by the vanishing of

$$\{\ f_i(x_1, \ldots, x_n) \mid i = 1, \ldots, l\ \} \cup \{\ g_j(x_{n+1}, \ldots, x_{n+m}) \mid j = 1, \ldots, k\ \} \subseteq k[x_1, \ldots, x_{n+m}].$$

(c.) The map $k[V] \times k[W] \to k[V \times W]$ given by $(f, g) \mapsto f(x_1, \ldots, x_n)g(x_{n+1}, \ldots, x_{n+m})$ is bilinear, and induces a k-algebra homomorphism

$$\phi : k[V] \otimes_k k[W] \to k[V \times W].$$

To see that ϕ is surjective, observe that $k[V \times W]$ is generated by the restriction of the coordinate functions on $\mathbb{A}^{n+m}$ to $V \times W$. These are all in the image of ϕ:

$$\phi(x_i \otimes 1) = x_i \quad \text{for} \quad 1 \le i \le n;$$

$$\phi(1 \otimes x_j) = x_j \quad \text{for} \quad n + 1 \le j \le n + m.$$

To see that ϕ is injective, consider $k[V] \otimes_k k[W]$ and $k[V \times W]$ as (probably infinite-dimensional) k-vector spaces. Let $\{\alpha_i\}_{i \in I}$ be a basis for $k[V]$, and $\{\beta_j\}_{j \in J}$ a basis for $k[W]$. Suppose that $h = \sum_{i,j} c_{ij}\alpha_i \otimes \beta_j$ is in the kernel of ϕ. This means for all $(x_1, \ldots, x_{n+m}) \in \mathbb{A}^{n+m}$,

$$\sum_{i,j} c_{ij}\, \alpha_i(x_1, \ldots, x_n) \cdot \beta_j(x_{n+1}, \ldots, x_{n+m}) = 0.$$

In particular, for any fixed $(x_1, \ldots, x_n) \in \mathbb{A}^n$, we have

$$\sum_{i,j} c_{ij}\, \alpha_i(x_1, \ldots, x_n) \cdot \beta_j = 0 \in k[W].$$

Since the β_j are linearly independent, for each j we must have

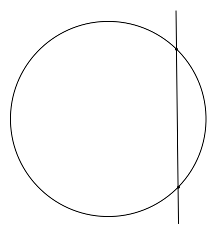
$$\sum_i c_{ij}\alpha_i(x_1,\ldots,x_n) = 0.$$

But $(x_1,\ldots,x_n)$ can be chosen arbitrarily, so we see $\sum_i c_{ij}\alpha_i = 0 \in k[V]$. Again, as the α_i are linearly independent, we must have $c_{ij} = 0$ for all $i \in I$ and $j \in J$. This shows $h = 0$, so ϕ is an isomorphism.

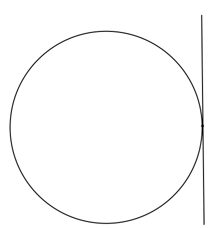
Problem 4. (a.) By the Nullstellensatz, we need to identify $\mathbb{I}(V_c) = \sqrt{\langle x-c, y^2-(1-c^2)\rangle}$. First, suppose that $c \neq \pm 1$. Then $y^2-(1-c^2)$ splits into two distinct linear factors (every nonzero element of k has two distinct square roots as $\operatorname{char}(k) \neq 2$). Thus, the ring

$$k[x,y]/\langle x-c, y^2-(1-x^2)\rangle \simeq k[y]/\langle y^2-(1-c^2)\rangle$$

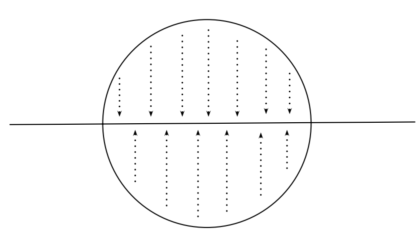
is reduced, and $\sqrt{\langle x-c, y^2-(1-c^2)\rangle} = \langle x-c, y^2-(1-c^2)\rangle$. Geometrically, the intersection consists of two points.



Next, suppose $c = \pm 1$. Then $y^2-(1-c^2) = y^2$, and it is easy to that $\sqrt{\langle x-c, y^2-(1-c^2)\rangle} = \langle x-c, y\rangle$. In this case, the intersection consists of a single point, at which the line $x = c$ is tangent to the circle $x^2+y^2 = 1$.



(b.) In terms of the projection map $\mathbb{A}^2 \to \mathbb{A}^1$ sending $(x,y) \mapsto x$, V_c is simply the fiber over $c \in \mathbb{A}^1$.



The special values $c = \pm 1$ are ramification points because they are the only ones for which there is a single point in the fiber; otherwise, there are two. In some sense, however, we should think of the fiber over $c = \pm 1$ as two points coming together into a "double point." Although not a variety, this is an example of geometric object called a scheme. The coordinate ring of the fiber over $c \neq \pm 1$ is $k[x,y]/\langle x-c, y^2-(1-c^2)\rangle$. As c approaches ± 1, this ring is "approaching" $k[x,y]/(x-1,y^2)$. So we would really like to say that $V_{\pm 1}$ is some geometric object whose coordinate ring is the *non-reduced* ring $k[x,y]/(x-1,y^2)$. Further, this geometric object can be thought of as a double point, or even more accurately, as the point $(1,0)$ with a first order tangent vector pointing vertically (do you see why the tangent points vertically, both algebraically and geometrically?).

(c.) In characteristic two, we have $x^2+y^2+1 = (x+y+1)^2$. In other words, the circle $x^2+y^2 = 1$ is really just the line $x+y = 1$ (although we would really like to think of it as a double line). In this case, V_c is always just the single point $(c, c+1)$, and $\mathbb{I}(V_c) = \langle x+c, y+c+1\rangle$. There are no special points (although, if we think of the circle as a double line, we would really like to think of every point as a special point, and the projection map as a degree two everywhere ramified cover).

(b.) I have a truly marvellous proof of this proposition which this margin is too narrow to contain.

Problem 5. (a.) If ϕ is an isomorphism, then the inverse mapping is given by polynomials $g_1, \ldots, g_n$ in variables $x_1, \ldots, x_n$ such that $g_i(f_1, \ldots, f_n) = x_i$ for all i. Let K be the Jacobian polynomial of the inverse, i.e. the determinant of the matrix $\partial g_i/\partial x_j$. Taking the partial derivative with respect to x_j gives

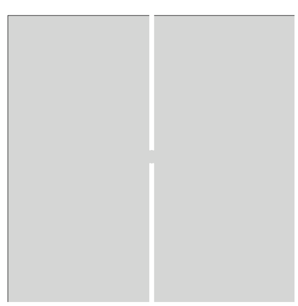
$$\sum_{k=1}^{n} \frac{\partial g_i}{\partial x_k}\frac{\partial f_k}{\partial x_j} = \delta_{ij}$$

for all i and j (δ_{ij} is the Kronecker delta). In other words, we have computed that the matrix product

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is the identity matrix. Taking determinants gives the equation $K \cdot J = 1$. Thus, since J is a unit in $k[x_1, \ldots, x_n]$, it is a nonzero constant.

Problem 6. The image of the regular map $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x,y) \mapsto (x, xy)$ is $(\mathbb{A}^2 \setminus \mathbb{V}(x)) \cup \{0\}$, i.e. it misses the y-axis except for the origin.



Since the image contains the open set $\mathbb{A}^2 \setminus \mathbb{V}(x)$, it is dense as $\mathbb{A}^2$ is irreducible. In particular, as the image is a proper subset of $\mathbb{A}^2$, it is not closed. Neither is the image open: the complement $\mathbb{V}(x) \setminus \{0\}$ is infinite, hence cannot be a proper closed subset of $\mathbb{V}(x) \simeq \mathbb{A}^1$.

Problem 7. The maps of coordinate rings are:

(Problem 2)	$\begin{array}{ccc} \mathbb{A}^1 & \to & \mathbb{A}^3 \\ k[t] & \leftarrow & k[x,y,z] \\ t & \leftarrow\!\shortmid & x \\ t^2 & \leftarrow\!\shortmid & y \\ t^3 & \leftarrow\!\shortmid & z \end{array}$	or	$\begin{array}{ccc} \mathbb{A}^1 & \to & V \\ k[t] & \leftarrow & \frac{k[x,y,z]}{\langle x^3-z, x^2-y\rangle} \\ t & \leftarrow\!\shortmid & x \\ t^2 & \leftarrow\!\shortmid & y \\ t^3 & \leftarrow\!\shortmid & z \end{array}$
(Problem 4 (b.))	$\begin{array}{ccc} \mathbb{A}^2 & \to & \mathbb{A}^1 \\ k[x,y] & \leftarrow & k[x] \\ x & \leftarrow\!\shortmid & x \end{array}$	or	$\begin{array}{ccc} \mathbb{V}(x^2+y^2-1) & \to & \mathbb{A}^1 \\ \frac{k[x,y]}{\langle x^2+y^2-1\rangle} & \leftarrow & k[x] \\ x & \leftarrow\!\shortmid & x \end{array}$
(Problem 5)	$\begin{array}{ccc} \mathbb{A}^n & \to & \mathbb{A}^n \\ k[x_1,\dots,x_n] & \leftarrow & k[x_1,\dots,x_n] \\ f_i & \leftarrow\!\shortmid & x_i \end{array}$		
(Problem 6)	$\begin{array}{ccc} \mathbb{A}^2 & \to & \mathbb{A}^2 \\ k[x,y] & \leftarrow & k[x,y] \\ x & \leftarrow\!\shortmid & x \\ xy & \leftarrow\!\shortmid & y \end{array}$		