MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 2 SOLUTIONS

Problem 1. (a.) Suppose V is a closed subvariety of $\mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$, and W is a closed subvariety of $\mathbb{A}^m$ with coordinates $y^1, \ldots, y^m$. Let $\phi : V \to W$ be a regular map. We need to show that ϕ is continuous in the Zariski topology. Since any regular map $V \to W$ extends to a regular map $\mathbb{A}^n \to \mathbb{A}^m$, we may assume $V = \mathbb{A}^n$ and $W = \mathbb{A}^m$. Let $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be the component functions of ϕ. Consider an arbitrary closed subset $Z = \mathbb{V}(g_1, \ldots, g_k) \subset \mathbb{A}^m$, where $g_1, \ldots, g_k \in k[y_1, \ldots, y_m]$. Then it follows that $\phi^{-1}(Z) = \mathbb{V}\left(g_1(f_1, \ldots, f_m), \ldots, g_k(f_1, \ldots, f_m)\right)$, hence $\phi^{-1}(Z)$ is closed and ϕ is continuous.

(b.) Consider $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto \bar{z}$. The Zariski topology on $\mathbb{C}$ is simply the finite complement topology. Since this map is bijective, it is certainly continuous in the Zariski topology (the inverse image of a finite set is finite). However, it is not regular as $\bar{z}$ is not a polynomial function!

Problem 2. (a.) Suppose first F is dominant, and $g \in \ker(F^*)$. Then $g \circ F$ is identically zero on V, i.e. $g|_{F(V)}$ is identically zero. In particular, $F(V) \subseteq \mathbb{V}(g)$, so $\mathbb{V}(g) = W$ as $F(V)$ is dense in W by assumption. In other words, $g = 0$ so F^* is injective.

Conversely, supose F^* is injective, and $g \in k[W]$ such that $g|_{F(V)}$is identically zero. Then it follose that $g \circ F = F^*(g) = 0$, hence $g = 0$ and $\mathbb{V}(g) = W$. Thus $\overline{F(V)} = W$ and F is dominant.

(b.) First, suppose F is a closed embedding. Since $W \subseteq \mathbb{A}^m$, we may assume $W = \mathbb{A}^m$ with coordinates $y_1, \ldots, y_m$. By assumption, $F(V)$ is closed in $\mathbb{A}^m$ and F induces an isomorphism between V and $F(V)$. Let $I = \mathbb{I}(F(V))$, so that the coordinate ring of $F(V)$ is $k[y_1, \ldots, y_m]/I$. The commutative diagram

$$V \xrightarrow{F} \mathbb{A}^m, \qquad V \xrightarrow{\simeq} F(V) \xhookrightarrow{\subseteq} \mathbb{A}^m$$

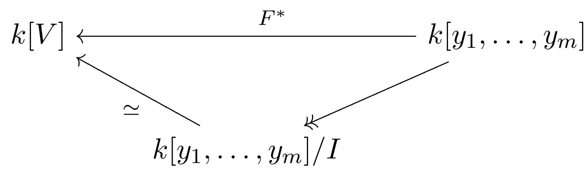
corresponds to the morphisms of coordinate rings

$$k[V] \xleftarrow{F^*} k[y_1, \ldots, y_m], \qquad k[y_1, \ldots, y_m] \twoheadrightarrow k[y_1, \ldots, y_m]/I \xrightarrow{\simeq} k[V]$$

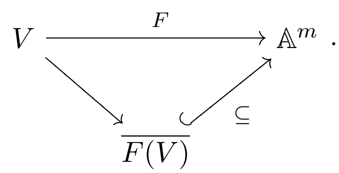
whence it is clear that F^* is surjective.

Conversely, suppose F^* is surjective. Again, we may assume $W = \mathbb{A}^m$. Let $I = \ker(F^*)$. Since $k[V]$ is reduced, I is a radical ideal. It follows from the definition of the Zariski topology that

$\mathbb{V}(I) = \overline{F(V)} \subseteq \mathbb{A}^m$. The commutative diagram of coordinate rings



corresponds to regular maps



It follows that F induces an isomorphism between V and $\overline{F(V)}$, since it induces an isomorphism on coordinate rings. In particular, $F(V)$ is closed and F is a closed embedding.

Problem 3. (a.) Every open set U of V can be written in the form $U = V \setminus \mathbb{V}(J)$ for some ideal $J \subseteq k[V]$. Thus,

$$U = V \setminus \mathbb{V}(J) = V \setminus \bigcap_{f \in J} \mathbb{V}(f) = \bigcup_{f \in J} V \setminus \mathbb{V}(f) = \bigcup_{f \in J} \mathcal{U}_f$$

is a union of basic open sets.

(b.) Suppose that $\{U_i\}_{i \in \mathscr{I}}$ is an open cover of V, with $U_i = V \setminus \mathbb{V}(J_i)$ for ideals $J_i \subseteq k[V]$. Then

$$V = \bigcup_{i \in \mathscr{I}} U_i = \bigcup_{i \in \mathscr{I}} V \setminus \mathbb{V}(J_i) = V \setminus \bigcap_{i \in \mathscr{I}} \mathbb{V}(J_i) = V \setminus \mathbb{V}(\sum_{i \in \mathscr{I}} J_i),$$

so that $\mathbb{V}(\sum_{i \in \mathscr{I}} J_i) = \emptyset$ and thus $\sum_{i \in \mathscr{I}} J_i = k[V]$. In particular, there is an equation of the form

$$a_1 f_1 + \cdots + a_m f_m = 1$$

where $a_j \in k[V]$ and $f_j \in J_{i_j}$, so that $\sum_{j=1}^m J_{i_j} = k[V]$. It follows that

$$V = V \setminus \emptyset = V \setminus \mathbb{V}(\sum_{j=1}^{m} J_{i_j}) = V \setminus \bigcap_{j=1}^{m} \mathbb{V}(J_{i_j}) = \bigcup_{j=1}^{m} V \setminus \mathbb{V}(J_{i_j}) = \bigcup_{j=1}^{m} U_{i_j}$$

so $\{U_{i_j}\}_{j=1}^m$ is a finite subcover.

(c.) Note that, since the individual points of a variety are closed, any finite collection of points is discrete and hence Hausdorff. We will show these are the only Hausdorff varieties.

Suppose first that V is an irreducible variety. Then any two nonempty open subsets must have nonempty intersection. Indeed, assume by way of contradiction that U_1 and U_2 are nonempty open subsets such that $U_1 \cap U_2 = \emptyset$. Then

$$V = V \setminus (U_1 \cap U_2) = (V \setminus U_1) \cup (V \setminus U_2)$$

realizes V as a union of proper closed subsets, which contradicts the very definition of irreducibility.

Since any Hausdorff topological space with at least two points must contain a pair of disjoint nonempty open subsets, we conclude from above that the only Hausdorff irreducible variety is a single point. More genearlly, if V is Hausdorff but not necessarily irreducible, then each of its irreducible components must be Hausdorff. Thus, we conclude that V must be a finite collection of points.

Problem 4. (a.) Let $x_1, \dots, x_n$ be the coordinates on $\mathbb{A}^n$, and z the coordinate on $\mathbb{A}^1$. The map

$$\begin{aligned} \phi : \mathcal{U}_f &\to \mathbb{A}^n \times \mathbb{A}^1 \\ p &\mapsto (p, \frac{1}{f(p)}) \end{aligned}$$

is clearly injective, since projection onto the first factor gives a left inverse. Suppose $\mathbb{I}(V) = \langle g_1, \dots, g_r \rangle$ for $g_1, \dots, g_r \in k[x_1, \dots, x_n]$. It is straightforward to check that the image of ϕ is the algebraic subset

$$\mathbb{V}(g_1(x_1, \dots, x_n), \dots, g_r(x_1, \dots, x_n), z \cdot f(x_1, \dots, x_n) - 1)$$

of $\mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1$ with coordinates $x_1, \dots, x_n, z$.

(b.) A closed set $C \subseteq V$ has the form $\mathbb{V}(g_1, \dots, g_r, h_1, \dots, h_s)$ for some $h_1, \dots, h_s \in k[x_1, \dots, x_n]$. Then $\phi(C) = \mathbb{V}(g_1, \dots, g_r, h_1, \dots, h_s, z \cdot f - 1)$ is also closed, so we need only check that ϕ is continuous.

Consider the closed subset $D = \mathbb{V}(u_1, \dots, u_t) \subset \mathbb{A}^{n+1}$ defined by $u_1, \dots, u_t \in k[x_1, \dots, x_n, t]$. Let M be the largest degree of any of $u_1, \dots, u_t$. Then for each $i = 1, \dots, t$, we have that the rational function

$$v_i(x_1, \dots, x_n) := (f(x_1, \dots, x_n))^M \cdot u_i \left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)} \right)$$

is actually a polynomial in $x_1, \dots, x_n$. Since f does not vanish along $\mathcal{U}_f$, it is easy to see that $\phi^{-1}(D) = \mathbb{V}(v_1, \dots, v_t) \cap \mathcal{U}_f$. Thus, ϕ is a homeomorphism onto its image.

(c.) We have that

$$k[x_1, \dots, x_n, z]/\langle g_1, \dots, g_r, z \cdot f - 1 \rangle \simeq k[V][z]/\langle z \cdot f - 1 \rangle \simeq k[V][\frac{1}{f}].$$

Since $k[V]$ is reduced, so is $k[V][\frac{1}{f}]$. Thus, $\langle g_1, \dots, g_r, z \cdot f - 1 \rangle$ is a radical ideal, $\mathbb{I}(\phi(V)) = \langle g_1, \dots, g_r, z \cdot f - 1 \rangle$, and $k[\phi(V)] = k[V][\frac{1}{f}]$.

(d.) Suppose U is open in V. Since the open sets of the form $\mathcal{U}_f$ for $f \in k[V]$ form a basis for the topology of V, we know that the basis elements which are contained in U form a basis for the subspace topology on U. From above, each of these can be identified with an affine algebraic subset of $\mathbb{A}^n \times \mathbb{A}^1$.

Problem 5. (a.) If the characteristic of k is two, then $x^2 + y^2 - 1 = (x + y - 1)^2$. The "circle" $\mathbb{V}(x^2 + y^2 - 1)$ is really the "line" $\mathbb{V}(x + y - 1)$. In particular, on this set we have $x = 1 - y$, and thus

$$\frac{1-y}{x} = \frac{x}{x} = 1$$

as rational functions. In particular, $\frac{1-y}{x}$ is regular on the entire "circle."

Alternatively, if the characteristic of k is not two, then one can show that x^2+y^2-1 is irreducible in $k[x,y]$. The rational function $\frac{1-y}{x}$ is manifestly regular at all points of the "circle" other than $(0,\pm1)$, where x vanishes. At $(0,1)$, $\frac{1-y}{x}$ is regular since it can be written $\frac{x}{1+y}$. Suppose, by way of contradiction, that it is also regular at $(0,-1)$. Then it would be regular on the entire "circle," and so it would agree with (the restriction of) a polynomial $p \in k[x,y]$. So we would have

$$\frac{x}{1+y} = p(x,y) \quad \text{or} \quad (1+y)\cdot p(x,y) = x$$

on a nonempty open subset of the "circle." Since the "circle" is irreducible, we must have that $(1+y)\cdot p(x,y) = x$ on its entirety. This implies that there is an equation of the form

$$x - (1+y)\cdot p(x,y) = (x^2+y^2-1)\cdot q(x,y)$$

in $k[x,y]$. Plugging in $y=1$ gives an equation of the form

$$x = x^2 \cdot q(x,0)$$

in the polynomial ring $k[x]$, an obvious absurdity. Thus, we conclude $\frac{1-y}{x}$ is regular everywhere on the "circle" except $(0,-1)$.

(b.) The rational function $\frac{y}{x}$ is regular except at $(0,0)$, the only point where x vanishes. To show this, we use the same trick as above. If it were, then it would agree with a polynomial $p(x,y)$ on the curve, which would mean that

$$y - x\cdot p(x,y) \in \langle y^2 - x^2 - x^3 \rangle$$

in the polynomial ring $k[x,y]$ (as $y^2-x^2-x^3$ is irreducible). Substituting $x=0$, this says $y \in \langle y^2 \rangle$ in $k[y]$, which is absurd.

Problem 6. (a.) Let $n = \dim V - 1, m = \dim W - 1$. Choose a basis $e_1, \ldots, e_{m+1}$ of W and extend it to a basis $e_1, \ldots, e_{m+1}, e_{m+2}, \ldots, e_{n+1}$ of V. Identify V with $\mathbb{A}^{n+1}$ and W with $\mathbb{A}^{m+1} \subset \mathbb{A}^{n+1}$ in the natural way with respect to this basis. Then $\mathbb{P}(V)$ corresponds to lines through the origin in $\mathbb{A}^{n+1}$, and $\mathbb{P}(W)$ corresponds to those lines which lie in the hyperplanes

$$x_{m+2} = x_{m+3} = \cdots = x_{n+1} = 0.$$

Thus $\mathbb{P}(W) = \mathbb{V}(x_{m+2}, \ldots, x_{n+1}) = \mathbb{V}(x_{m+2}) \cap \cdots \cap \mathbb{V}(x_{n+1}) \subset \mathbb{P}(V)$. This is the intersection of $c = n - m = \operatorname{codim} \mathbb{P}(W)$ hyperplanes.

(b.) From the definition, it is obvious that $\mathbb{P}(W_1) \cap \mathbb{P}(W_2) = \mathbb{P}(W_1 \cap W_2)$. It is also clear that the codimension of W_i in V is the same as the codimension of $\mathbb{P}(W_i)$ in $\mathbb{P}(V)$. Now we can just use the elementary fact that $\operatorname{codim} W_1 \cap W_2 \leq \operatorname{codim} W_1 + \operatorname{codim} W_2$ for these vector subspaces to see that the same applies to projective linear subspaces.

Problem 7. The answers to this problem are entirely contained in Hartshorne's classic text. We will not attempt to improve upon his exposition. However, should you find them difficult to digest, we would encourage you to return to this problem as the semester progresses. Consider the following quote of Andrew Wiles in which he compares doing mathematics with exploring a dark mansion:

> You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of, and couldn't exist without, the many months of stumbling around in the dark that precede them.