MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 3 SOLUTIONS

Problem 1. The rational function $\phi = \frac{x}{z}$ on $\mathbb{V}(z^2 - xy)$ is certainly regular where $z \neq 0$. Further, since we have $\frac{x}{z} = \frac{z}{y}$ as well, ϕ is also regular whenever $y \neq 0$. By way of contradiction, assume for the moment that ϕ is regular at a point $(\lambda, 0, 0) \in \mathbb{V}(z^2 - xy)$, where $\lambda \neq 0$. Then $\phi \cdot z = x$ in a neighborhood of $(\lambda, 0, 0)$, and evaluating these functions at this point gives the nonsensical equation $0 = \phi(\lambda, 0, 0) \cdot 0 = \lambda$. Thus, ϕ is not regular at $(\lambda, 0, 0)$. Since the locus of indeterminacy of ϕ is closed, we also conclude that ϕ is not regular at $(0, 0, 0)$. It follows that $\mathbb{V}(z^2 - xy) \setminus \mathbb{V}(x, y) = \{\ (x, y, z) \in \mathbb{V}(z^2 - xy) \mid y \neq 0 \ \text{ or } \ z \neq 0\ \}$ is the domain of defintion of ϕ.

Problem 2. (a.) Suppose I is homogeneous, and pick homogeneous generators $\{h_\alpha\}$ of I. Let d_α be the degree of h_α. If $f \in I$, then we can write $f = \sum g_\alpha h_\alpha$ for some $g_\alpha \in k[X_0, \ldots, X_n]$. Write each g_α in terms of its homogeneous components, i.e. write $g_\alpha = \sum g_{\alpha,k}$ where the degree of $g_{\alpha,k}$ is k. The homogeneous component of f in degree d is $\sum g_{\alpha,d-d_\alpha} h_\alpha$, which is certainly an element of I.

Conversely, suppose I has the property that it contains the homogeneous components of each of its elements. It is straightforward to check that the homogeneous components of a set of generators for I also generate. In particular, I is homogeneous.

(b.) If $\{f_\alpha\}$ and $\{g_\beta\}$ are sets of homogeneous generators for I and J, respectively, then $I + J$ is generated by the homogeneous polynomials $\{f_\alpha + g_\beta\}$. Similarly, IJ is generated by the homogeneous polynomials $\{f_\alpha \cdot g_\beta\}$. Thus, $I + J$ and IJ are homogeneous. If $f \in I \cap J$, then its homogeneous components are in both I and J, i.e. they are in $I \cap J$. By part (a.), $I \cap J$ is homogeneous.

Lastly, suppose $f \in \sqrt{I}$. Write $f = f_0 + \cdots + f_n$ in terms of its homogeneous components. Assume, by way of contradiction, $f_j \notin \sqrt{I}$ for some j. Without loss of generality, we may assume $j = n$ (by replacing f with $f - (f_{j+1} + \cdots + f_n)$ in case $f_{j+1}, \ldots, f_n \in \sqrt{I}$). There is some N such that $f^N \in I$. The homogeneous component of f^N in degree nN is $(f_n)^N$. As I is homogeneous, we must have $(f_n)^N \in I$. It follows that $f_n \in \sqrt{I}$, which is a contradiction. Thus, all the f_j must have been in I, and I is homogeneous by part (a.).

(c.) The projective version Hilbert's Nullstellensatz may be stated as:

> The projective subvarieties of $\mathbb{P}^n$ stand in one-to-one correspondence with the radical ideals of the ring $k[x_0, \ldots, x_n]$ that admit a homogeneous set of generators, with the exception of $\langle x_0, \ldots, x_n \rangle$ (which defines the origin in the space k^{n+1}).

This theorem follows easily from the affine version by considering the cone in k^{n+1} over a projective subvariety of $\mathbb{P}^n$. Let $\pi : k^{n+1} \setminus \{0\} \to k^{n+1} \setminus \{0\}/k^* \simeq \mathbb{P}^n$ be the quotient map. If V is a closed subvariety of $\mathbb{P}^n$, then the cone $C(V)$ over V is the closure of $\pi^{-1}(V)$ in k^{n+1}. It is easy to see that this is precisely $\pi^{-1}(V) \cup \{0\}$: if $\{h_\alpha\}$ are homogeneous equations cutting out V, then these same equations cut out the cone over V in k^{n+1}. From the affine Nullstellensatz, it follows immediately that $\mathbb{I}(C(V)) = \sqrt{\langle h_\alpha \rangle}$ (which we know is homogeneous from above). The remainder of the statement is easy to check. One must take special care, however, as both $\langle x_0, \dots, x_n \rangle$ and $\langle 1 \rangle$ define the empty set in $\mathbb{P}^n$.

Problem 3. (a.) Suppose X is a nonsingular quadric in $\mathbb{P}^n$. Let Q be the associated degree two homogeneous polynomial. Consider $\mathbb{P}^n$ as $\mathbb{P}(V)$ for an abstract vector space V of dimension $n+1$ over k, and Q as a quadratic form on V. By assumption, the symmetric bilinear form

$$B(v, w) = Q(v + w) - Q(v) - Q(w)$$

is nondegenerate. Since k does not have characteristic 2, we have $Q(v) = \frac{1}{2}B(v, v)$. Because k is algebraically closed, we may use the Gram-Schmidt process to find a basis for V in which the matrix of B is the identity matrix. This change of basis corresponds to a linear change of coordinates on $\mathbb{P}^n$ which takes X to $\mathbb{V}(x_0^2 + \cdots + x_n^2)$. Thus, all nonsingular quadrics are projectively equivalent to $\mathbb{V}(x_0^2 + \cdots + x_n^2)$, and hence to each other.

(b.) Let x, y, z be the homogeneous coordinates of $\mathbb{P}^2$. Consider the line $X_1 = \mathbb{V}(x)$ and the nonsingular quadric $X_2 = \mathbb{V}(xy - z^2)$. Both X_1 and X_2 are isomorphic to $\mathbb{P}^1$ (note that X_2 is the image of the second Veronese embedding of $\mathbb{P}^1$). However, since a linear change of coordinates on $\mathbb{P}^2$ must preserve the degrees of homogeneous equations, they are not projectively equivalent.

Problem 4. Recall that every quasi-projective variety X in $\mathbb{P}^n$ is open in its closure $\overline{X}$ in $\mathbb{P}^n$. Thus, it is sufficient to show that X is irreducible if and only if $\overline{X}$ is irreducible. Suppose first that $\overline{X}$ is irreducible. If U_1 and U_2 are nonempty open subsets of X, then they are also open in $\overline{X}$. Hence, $U_1 \cap U_2$ is nonempty as $\overline{X}$ is irreducible, and it follows that X is also irreducible.

Conversely, suppose that X is irreducible. Suppose V_1 and V_2 are nonempty open subsets of $\overline{X}$. Every open subset of $\overline{X}$ intersects X nontrivially, so $V_1 \cap X$ and $V_2 \cap X$ are nonempty open subsets of X. Thus, as X is irreducible, $X \cap V_1$ and $X \cap V_2$ intersect nontrivially. It follows that $V_1 \cap V_2$ is nonempty, so $\overline{X}$ is irreducible.

Problem 5. (a.) There is a bijective correspondence between linear subspaces of $\mathbb{P}^n$ and vector subspaces of k^{n+1}. If $\widetilde{\Lambda} \subset k^{n+1}$ is the vector subspace corresponding to Λ and $\widetilde{x}$ is the one-dimensional subspace corresponding to x, then $\widetilde{\Lambda} + \widetilde{x}$ is the unique vector subspace of dimension $\dim \Lambda + 2$ containing $\widetilde{\Lambda}$ and $\widetilde{x}$. Hence $\mathbb{P}(\widetilde{\Lambda} + \widetilde{x})$ is the unique dimension $\dim \Lambda + 1$ linear subspace of $\mathbb{P}^n$ containing Λ and x. Furthermore, $(\widetilde{V} + \widetilde{x}) \cap \widetilde{L}$ is a one-dimensional subspace of k^{n+1}, so $\mathbb{P}(\widetilde{V} + \widetilde{x}) \cap L$ is a single point in $\mathbb{P}^{n+1}$.

(b.) We can change coordinates so that $\Lambda = \mathbb{V}(x_0, \dots, x_m)$ and $L = \mathbb{V}(x_{m+1}, \dots, x_n)$. It is then clear that $\pi_{\Lambda,L} : [x_0 : \cdots : x_n] \mapsto [x_0 : \ldots : x_m : 0 : \ldots : 0]$. In particular, it is regular on $\mathbb{P}^n - \Lambda$.

(c.) Let's use the coordinates chosen in part (b). Projecting from $[0 : \ldots : 0 : 1]$ gives us $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_{n-1} : 0]$. Composing now with the projection from $[0 : \ldots : 0 : 1 : 0]$ gives $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_{n-2} : 0 : 0]$. Continuing in this fashion, we see that π_Λ is the composition of m projections from points.

(d.) Since L is disjoint from Λ, we see that $\pi_{\Lambda,L'}$ restricted to L is a regular morphism from L to L' given by linear polynomials, and similarly for $\pi_{\Lambda,L}$ restricted to L'. It is easy to check that these are mutually inverse, and that $\pi_{\Lambda,L}(V)$ corresponds to $\pi_{\Lambda,L'}(V)$ under $\pi_{\Lambda,L'}$ restricted to L.[1]

(e.) Let V be two points of $\mathbb{P}^n$. Let ℓ be the line joining the two points of V. Projecting from a point of $\ell \setminus V$ yields a single point, while projection from a point not on ℓ gives two distinct points.[2]

Problem 6. (a.) Th map $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ is given by $[s : t] \mapsto [s^3 : s^2 t : st^2 : t^3]$. If Y_0, Y_1 are the homogeneous coordinates on $\mathbb{P}^1$ and X_0, X_1, X_2, X_3 are the homogeneous coordinates on $\mathbb{P}^3$, then ν maps $\mathbb{P}^1 \setminus \mathbb{V}(Y_0) \simeq \mathbb{A}^1$ into $\mathbb{P}^3 \setminus \mathbb{V}(X_0) \simeq \mathbb{A}^3$ via $t \mapsto (t, t^2, t^3)$.

(b.) One easily checks that C lies on

Q_0: $(s^3)(st^2) - (s^2 t)^2 = 0$
Q_1: $(s^3)(t^3) - (s^2 t)(st^2) = 0$
Q_2: $(s^2 t)(t^3) - (st^2)^2 = 0$

Conversely, suppose $[w : x : y : z] \in Q_0 \cap Q_1 \cap Q_2$. Then $wy = x^2$, $wz = xy$, and $xz = y^2$. If either $w = 0$ or $z = 0$, these relations imply $x = y = 0$. In this case we have $[w : 0 : 0 : 0] = [1 : 0 : 0 : 0]$ or $[0 : 0 : 0 : z] = [0 : 0 : 0 : 1]$. These are the image of $[1 : 0]$ and $[0 : 1]$ respectively. Otherwise, $w \neq 0$ and $z \neq 0$. Then $wz = xy$ implies that $x \neq 0$ and $y \neq 0$. So we can multiply by xy to get $[w : x : y : z] = [wxy : x^2 y : xy^2 : xyz] = [x^3 : x^2 y : xy^2 : y^3]$, which is mapped to by $[x : y]$.

(c.) It is clear that

(i) $Q_0 \cap Q_1 = \mathbb{V}(X_0 X_2 - X_1^2, X_0 X_3 - X_1 X_2) \supseteq C \cup \mathbb{V}(X_0, X_1)$
(ii) $Q_1 \cap Q_2 = \mathbb{V}(X_0 X_3 - X_1 X_2, X_1 X_3 - X_2^2) \supseteq C \cup \mathbb{V}(X_2, X_3)$
(iii) $Q_2 \cap Q_0 = \mathbb{V}(X_1 X_3 - X_2^2, X_0 X_2 - X_1^2) \supseteq C \cup \mathbb{V}(X_1, X_2)$

In each case the containment is clear since the reverse containment of ideals holds, e.g. $X_0 X_2 - X_1^2 \in (X_0, X_1)$. Now we prove containment in the other direction. For each case suppose $[w : x : y : z] \in (Q_i \cap Q_j) \setminus C$.

[1]It is tempting here to prove this by using a linear automorphism ϕ of $\mathbb{P}^n$ fixing Λ and maping L to L'. However, ϕ almost certainly does not fix V, and hence this only gives an isomorphism from L to L' which takes $\pi_{\Lambda,L}(V)$ to $\pi_{\Lambda,L'}(\phi(V))$ (which, again, is probably not $\pi_{\Lambda,L'}(V)$ as ϕ does not fix V).
[2]Note, by the previous part, it does not matter what we project onto.

(i) We have $wy = x^2$, $wz = xy$, and $xz \neq y^2$. The first two relations give $wy^2 = wxz \implies w(y^2 - xz) = 0$, so $w = 0$ because $xz \neq y^2$. Then the first relation gives $x = 0$. Thus $[w : x : y : z] \in \mathbb{V}(X_0, X_1)$.

(ii) We have $wz = xy$, $xz = y^2$, and $wy \neq x^2$. The first two relations give $wyz = x^2z \implies z(wy - x^2) = 0$, so $z = 0$ because $wy \neq x^2$. Then the second relation gives $y = 0$. Thus $[w : x : y : z] \in \mathbb{V}(X_2, X_3)$.

(iii) We have $xz = y^2$, $wy = x^2$, and $wz \neq xy$. The first two relations give $wxz = x^2y \implies x(wz - xy) = 0$, so $x = 0$ because $wz \neq xy$. Then the first relation gives $y = 0$. Thus $[w : x : y : z] \in \mathbb{V}(X_1, X_2)$.

(d.) We seek two homogeneous polynomials P, Q such that $C = \mathbb{V}(P, Q) = \mathbb{V}(P) \cap \mathbb{V}(Q)$. So, I claim that

$$C = \mathbb{V}(X_2^2 - X_1X_3, X_1^3 - 2X_0X_1X_2 + X_0^2X_3) =: V$$

It's clear that $C \subset V$, since if $[s^3 : s^2t : st^2 : t^3] \in C$, then $(st^2)^2 - t^3s^2t = 0$ and $(s^2t)^3 - 2(s^3)(s^2t)(st^2) + (s^3)^2(t^3) = 0$, so $[s^3 : s^2t : st^2 : t^3] \in V$.

Now, suppose that $[x_0 : x_1 : x_2 : x_3] \in V$. Then, we have $x_2^2 - x_1x_3 = 0$ and $x_1^3 - 2x_0x_1x_2 + x_0^2x_3 = 0$. If $x_1 \neq 0$, then without loss of generality, $x_1 = 1$. Then, our equations read $x_2^2 = x_3$ and $1 - 2x_0x_2 + x_0^2x_3 = 0$, so substituting the first equation into the second equation, we have $0 = 1 - 2x_0x_2 + x_0^2x_2^2 = (1 - x_0x_2)^2$. So, neither of x_0, x_2 are 0, and $x_0 = \frac{1}{x_2}$. So, we see that $[x_0 : x_1 : x_2 : x_3] = [\frac{1}{x_2} : 1 : x_2 : x_2^2] = [1 : x_2 : x_2^2 : x_2^3] = \nu([1 : x_2])$, so $[x_0 : x_1 : x_2 : x_3] \in C$. If $x_1 = 0$, then from the first equation $x_2 = 0$, and so the second equation reduces to $x_0^2x_3 = 0$. Both of x_0, x_3 cannot vanish because not all homogeneous coordinates can vanish. So, if $x_0 = 0$, then $[x_0 : x_1 : x_2 : x_3] = [0 : 0 : 0 : x_3] = \nu([0 : x_3])$, and if $x_3 = 0$, then $[x_0 : x_1 : x_2 : x_3] = [x_0 : 0 : 0 : 0] = \nu([x_0 : 0])$. In either case, $[x_0 : x_1 : x_2 : x_3] \in C$, so $C \subset V$, and thus $C = V$, and C is the intersection of two hypersurfaces, as required.

These two equations do not generate the full radical ideal of homogeneous polynomials that vanish on C. To see this note that $X_3X_0 - X_1X_2$ clearly vanishes on C, but $X_3X_0 - X_1X_2 \notin \langle X_2^2 - X_1X_3, X_1^3 - 2X_0X_1X_2 + X_0^2X_3 \rangle$. The only degree two polynomials in this ideal are scalar multiples of $X_2^2 - X_1X_3$, none of which are equal to $X_3X_0 - X_1X_2$.

(e.) The Segre 3-fold in $\mathbb{P}^5$ is the vanishing set of all 2×2 minors of

$$\begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix}$$

i.e. $\mathbb{V}(AE - BD, AF - CD, BF - CE) \subset \mathbb{P}^5$. The twisted cubic is $\mathbb{V}(X_0X_2 - X_1^2, X_0X_3 - X_1X_2, X_1X_3 - X_2^2) \subset \mathbb{P}^3$. We can embed $\mathbb{P}^3$ into $\mathbb{P}^5$ via a map φ which sends $[w : x : y : z] \mapsto [w : x : y : x : y : z]$. The image of $\mathbb{P}^3$ under φ is clearly the 3-plane $\mathbb{V}(B - D, C - E)$. It is trivial to check that the image of the twisted cubic under φ is $\mathbb{V}(AE - BD, AF - CD, BF - CE) \cap \mathbb{V}(B - D, C - E)$, i.e. the intersection of the Segre 3-fold with this 3-plane.

Problem 7. (a.) Without loss of generality, we may assume V is an affine variety throughout this problem. Note that $k[V] \subset \mathcal{O}_{V,x} \subset k(V)$, so we only need to check $\mathcal{O}_{V,x}$ is closed under addition and multiplication. Suppose $\phi_1, \phi_2 \in \mathcal{O}_{V,x}$. Then for $i = 1, 2$ we have $\phi_i = \frac{p_i}{q_i}$ for regular functions $p_i, q_i \in k[V]$ with $q_i(x) \neq 0$. Since $q_1(x) \cdot q_2(x) \neq 0$ as well, it follows that $\phi_1 + \phi_2 = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}$ and $\phi_1 \cdot \phi_2 = \frac{p_1 p_2}{q_1 q_2}$ are also regular at x.

(b.) We will use the notation from part (a.). If $\phi_1(x) = 0$, then $(\phi_1 \cdot \phi_2)(x) = \phi_1(x) \cdot \phi_2(x) = 0$. If, in addition, $\phi_2(x) = 0$, we have $(\phi_1 + \phi_2)(x) = \phi_1(x) + \phi_2(x) = 0$. Thus, $\mathfrak{m}_x$ is an ideal. On the other hand, if $\phi_2(x) \neq 0$, we have $p_2(x) \neq 0$, so that $\frac{1}{\phi_2} = \frac{q_2}{p_2} \in \mathcal{O}_{V,x}$. Thus, $\mathcal{O}_{V,x}$ is local with maximal ideal $\mathfrak{m}_x$.

(c.) Since ϕ is defined in a neighborhood of x, we may simply write $\phi = \phi(x) + (\phi - \phi(x))$ and notice that $\phi - \phi(x) \in \mathfrak{m}_x$.

(d.) If $\phi \in \mathcal{O}_{V,x}$ and $\phi = \frac{p}{q}$ with $p, q \in k[V]$ and $q(x) \neq 0$. Then clearly $\phi \in \mathcal{O}_V(V \setminus \mathbb{V}(q))$. Since two rational functions are equal if and only if they agree on a non-empty open subset of V, it follows immediately that $\mathcal{O}_{V,x} = \varinjlim_{x \in U \subset V} \mathcal{O}_V(U)$.

(e.) By definition, the stalk of $\mathcal{O}_V$ at x is $\varinjlim_{x \in U \subset V} \mathcal{O}_V(U)$, which is $\mathcal{O}_{V,x}$ by part (d.). Since these are local rings by (b.), $\mathcal{O}_V$ is a sheaf of rings on V whose stalks are all local, and thus every irreducible variety has a natural structure of a locally ringed space.