

## MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 4 SOLUTIONS

**Problem 1.** (*Definition*  $\implies$  (a.)) Suppose  $\phi : X \rightarrow Y$  is a morphism, and  $x \in X$ . We may assume  $U$  is a neighborhood of  $x$  with  $\phi(U) \subseteq \mathcal{U}_0 \subseteq \mathbb{P}^m$  such that

$$\phi|_U : U \rightarrow Y \cap \mathcal{U}_0 \subseteq \mathcal{U}_0 \simeq \mathbb{A}^m$$

is given by

$$y \mapsto (\phi_1(y), \dots, \phi_m(y)) \quad \text{where} \quad \phi_1, \dots, \phi_m \in \mathcal{O}_X(U).$$

Shrinking  $U$  if necessary, we may write  $\phi_i(y) = \frac{p_i(y)}{q_i(y)}$  for all  $y \in U$ , where  $p_i, q_i$  are homogeneous polynomials of the same degree  $d_i$  in the coordinates  $x_0, \dots, x_n$  of  $\mathbb{P}^n$  with  $q_i(y) \neq 0$  for  $y \in U$ . Let  $Q = \prod_{i=1}^m q_i$ , which is homogeneous of degree  $d = \sum_{i=1}^m d_i$ . For each  $y \in U$ , we have  $Q(y) \neq 0$  and it follows that

$$\phi(y) = [1 : \phi_1(y) : \dots : \phi_m(y)] = [1 : \frac{p_1(y)}{q_1(y)} : \dots : \frac{p_m(y)}{q_m(y)}] = [Q(y) : (Q(y)\frac{p_1(y)}{q_1(y)}) : \dots : (Q(y)\frac{p_m(y)}{q_m(y)})].$$

Since  $F_i := Q \cdot \frac{p_i}{q_i} = p_i \cdot \prod_{j \neq i} q_j$  and  $F_0 = Q$  are homogeneous polynomials of degree  $d$ , we have shown that  $\phi$  satisfies (a.).

( (a.)  $\implies$  (b.)) Note first that  $\phi$  is certainly continuous: if  $Z = \mathbb{V}(f_1, \dots, f_k) \cap Y$  for homogeneous polynomials  $f_1, \dots, f_k$  in the coordinates  $y_0, \dots, y_m$  of  $\mathbb{P}^m$ , then  $\phi^{-1}(Z)$  locally looks like  $X \cap \mathbb{V}(f_1(F_0, \dots, F_m), \dots, f_k(F_0, \dots, F_m))$  for homogeneous polynomials  $F_0, \dots, F_m$  of the same degree on  $\mathbb{P}^n$ . Each  $f_j(F_0, \dots, F_m)$  is thus a homogeneous polynomial on  $\mathbb{P}^n$ , and being closed is a local property. Thus,  $\phi^{-1}(Z)$  is closed, and  $\phi$  is continuous.

Next, suppose that  $\psi \in \mathcal{O}_Y(V)$  for some open subset  $V$  of  $Y$ , and  $U$  is an open subset of  $X$  with  $\phi(U) \subseteq V$ . Let  $x \in U$ . Since the rational function  $\psi$  is regular at  $\phi(x)$ , we may write  $\psi(y) = \frac{p(y)}{q(y)}$  for all  $y$  in a neighborhood of  $\phi(x)$  and some homogeneous polynomials  $p, q$  on  $\mathbb{P}^m$  of the same degree with  $q(y) \neq 0$ . Thus, for all  $z$  in a neighborhood of  $x$  (since  $\phi$  is continuous), we may write  $\psi \circ \phi(z) = \frac{p(F_0(z), \dots, F_m(z))}{q(F_0(z), \dots, F_m(z))}$  for homogeneous polynomials  $F_0, \dots, F_m$  on  $\mathbb{P}^n$  of the same degree. Since  $p(F_0, \dots, F_m)$  and  $q(F_0, \dots, F_m)$  are homogeneous polynomials of the same degree on  $\mathbb{P}^n$ , it follows that  $\psi \circ \phi$  is regular at  $x$ . We conclude that  $\psi \circ \phi \in \mathcal{O}_X(U)$ .

Consider any affine covers  $\{U_\lambda\}$  of  $X$  and  $\{V_\mu\}$  of  $Y$  with each  $\phi(U_\lambda)$  contained in some  $V_\mu$ . From above, we see that pulling back under  $\phi$  induces a ring homomorphism  $\mathcal{O}_Y(V_\mu) \rightarrow \mathcal{O}_X(U_\lambda)$  whenever  $\phi(U_\lambda) \subseteq V_\mu$ . Under the equivalence of categories, we conclude that  $\phi|_{U_\lambda}$  is a regular map from  $U_\lambda$  to  $V_\mu$  (in the sense of affine varieties).

( (b.)  $\implies$  *definition* ) As was pointed out by various students, this is not a well-posed question. However, the point is that regular maps are *local* beasts. A correctly stated version of b can be found on the revised (and posted) problem set 4.

**Problem 2.** Following the hint, we first treat the case where  $H$  is defined by the vanishing of a linear polynomial on  $\mathbb{P}^n$ . After a linear change of coordinates, we may assume  $H = \mathbb{V}(x_0)$ . Then  $\mathbb{P}^n \setminus H = \mathcal{U}_0 \simeq \mathbb{A}^n$  is one of the affine open sets in the standard cover of  $\mathbb{P}^n$ .

Now suppose  $H = \mathbb{V}(f)$  where  $f$  is a homogeneous polynomial of degree  $d > 1$ . Write  $f = \sum_I c_I x^I$ , where  $c_I \in k$  and the sum ranges over all of the monomials  $x^I$  of degree  $d$  on  $\mathbb{P}^n$ . Let

$$\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+d}{d}-1}$$

$$[x_0 : \cdots : x_n] \mapsto [\cdots : x^I : \cdots]$$

be the  $d$ -th Veronese embedding. Let  $\dots, z_I, \dots$  be the coordinates on  $\mathbb{P}^{\binom{n+d}{d}-1}$ . Then we have  $f = F \circ \nu_d$ , where  $F = \sum_I c_I z_I$  is a linear polynomial on  $\mathbb{P}^{\binom{n+d}{d}-1}$ . Thus,  $\nu_d$  identifies  $\mathbb{P}^n \setminus H$  with a closed subset of  $\mathbb{P}^{\binom{n+d}{d}-1} \setminus \mathbb{V}(F) \simeq \mathbb{A}^{\binom{n+d}{d}-1}$ .

**Problem 3.** Consider  $\mathbb{P}^n$  as  $\mathbb{P}(V)$  for an  $(n+1)$ -dimensional vector space  $V$  over  $k$ . Then  $x_1, \dots, x_n \in \mathbb{P}(V)$  correspond to lines  $L_1, \dots, L_n$  in  $V$ . We can find at least one linear subspace  $W$  of  $V$  with  $\dim(W) = n$  and  $L_1 + \cdots + L_n \subseteq W$ , so that  $\mathbb{P}(W)$  is a hyperplane containing  $x_1, \dots, x_n$ . If the  $x_1, \dots, x_n$  are in general position, i.e.  $\dim(L_1 + \cdots + L_n) = n$ , then  $W = L_1 + \cdots + L_n$  is unique.

**Problem 4.** (a.) Let  $P = [p_0 : \cdots : p_n]$ , and let  $\dots, x^I, \dots$  be the monomials of degree  $d$  on  $\mathbb{P}^n$ . Denote by  $p^I$  the value of the monomial  $x^I$  at  $(p_0, \dots, p_n)$ . If  $Q_d = \sum_I a_I x^I$  with  $a_I \in k$  is a homogeneous polynomial of degree  $d$  defining the hypersurface  $X_d$  in  $\mathbb{P}^n$ , then  $P \in X_d$  if and only if

$$Q(p_0, \dots, p_n) = \sum_I a_I p^I = 0.$$

Since this is a linear relation on the coordinates  $\dots, a_I, \dots$  of  $\text{Sym}^d(V^*)$ , we see that the set of hypersurfaces of degree  $d$  through  $P$  is given by a hyperplane as desired.

(b.) Let  $q = \binom{d+2}{d} - 1$ , and  $\mathbb{P}^2 = \mathbb{P}(V)$ . Then  $\mathbb{P}(\text{Sym}^d(V^*)) \simeq \mathbb{P}^q$ . Fix  $p_1, \dots, p_q \in \mathbb{P}^2$ , and let  $H_1, \dots, H_q$  be the corresponding hyperplanes in  $\mathbb{P}^q$  as above. If  $p_1, \dots, p_q \in \mathbb{P}^2$  are sufficiently general, then  $H_1 \cap \cdots \cap H_q$  will be a single point in  $\mathbb{P}^q$ . Thus, there is a uniquely determined hypersurface of degree  $d$  through  $q$  general points in  $\mathbb{P}^2$ .

**Problem 5.** Let  $x, y$  be the coordinates on  $\mathbb{A}^2$ . Then  $X = \mathcal{U}_x \cup \mathcal{U}_y$ , where  $\mathcal{U}_x = \mathbb{A}^2 \setminus \mathbb{V}(x)$ , and similarly for  $\mathcal{U}_y$ . We have

$$\mathcal{O}_{\mathbb{A}^2}(X) = \mathcal{O}_{\mathbb{A}^2}(\mathcal{U}_x) \cap \mathcal{O}_{\mathbb{A}^2}(\mathcal{U}_y) = k[x, y] \left[ \frac{1}{x} \right] \cap k[x, y] \left[ \frac{1}{y} \right] = k[x, y].$$

This shows that  $X$  cannot be affine. If it were, the inclusion map of  $X$  into  $\mathbb{A}^2$  would induce an isomorphism of coordinate rings. From the equivalence of categories, we would have that the inclusion map was an isomorphism – but it is not even surjective!

**Problem 6.** (a.) Write  $X = \mathbb{V}(f_1, \dots, f_k)$ , where  $f_j \in k[x_0, \dots, x_n]$  is homogeneous of degree  $d_j$  for  $j = 1, \dots, k$ . Let  $d = \max\{d_1, \dots, d_k\}$ . Since  $\mathbb{V}(x_0, \dots, x_n) = \emptyset$ , we have  $X = \mathbb{V}(\{x_0^{d-d_j} f_j, \dots, x_n^{d-d_j} f_j\}_{j=1}^k)$ . Each  $x_i^{d-d_j} f_j$  is homogeneous of degree  $d$ .

(b.) From above, we can write  $X = \mathbb{V}(\{g_1, \dots, g_\ell\})$  where each  $g_j \in k[x_0, \dots, x_n]$  is homogeneous of degree  $d$ . Let  $\{x^\alpha\}_{\alpha \in \mathcal{A}}$  be the set of monomials of degree  $d$  in  $x_0, \dots, x_n$ , and write  $g_j = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$ . Consider the  $d$ -th Veronese embedding  $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^{N=\binom{n+d}{d}}$ . If we express the homogeneous coordinates of  $\mathbb{P}^N$  as  $\{z_\alpha\}_{\alpha \in \mathcal{A}}$ , we have that  $\nu_d([x_0 : \dots : x_n]) = [\dots : z_\alpha : \dots]$ . In particular, let  $h_j = \sum_{\alpha \in \mathcal{A}} c_\alpha z_\alpha$  and  $L = \mathbb{V}(h_1, \dots, h_\ell)$  the corresponding linear space. Since  $h_j \circ \nu_d = g_j$ , we have that  $\nu_d$  maps  $X$  isomorphically onto  $V_d \cap L$ , as desired.

(c.) Using the notation from the previous part, it is enough to see that  $V_d \cap L$  is defined by quadrics in  $L$ . This is clear:  $V_d$  is defined by quadrics in  $\mathbb{P}^N$ , and a quadric in  $\mathbb{P}^N$  restricts to a quadric on any linear subspace of  $\mathbb{P}^N$ .

**Problem 7.** (a.) From previous assignments, we know that linear subvarieties of  $\mathbb{P}(V)$  of dimension  $d$  have the form  $\mathbb{P}(W)$  for a uniquely determined  $(d+1)$ -dimensional vector subspace  $W$  of  $V$ . Thus, a line in  $\mathbb{P}(V)$  corresponds to two dimensional subspace of  $V$ , i.e. to a point in  $\mathbb{G}_2(V)$ . Similarly, a linear subvariety of dimension  $d$  in  $\mathbb{P}(V)$  corresponds to a  $d+1$  dimensional subspace of  $V$ , which is a point in  $\mathbb{G}_{d+1}(V)$ .

(b.) Pick a basis  $v_1 = (a_1^1, \dots, a_n^1), \dots, v_d = (a_1^d, \dots, a_n^d)$  of  $W$ . Then the subspace  $W$  corresponds to the row space of the  $d \times n$  matrix

$$\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^d & a_2^d & \cdots & a_n^d \end{pmatrix}.$$

In other words, the subspace  $W$  is the image of  $k^d$  under the linear mapping determined by this matrix. Since  $v_1, \dots, v_d$  are linearly independent, this matrix has full rank.

(c.) A different choice of basis for  $W$  would give a different matrix, but there would be some change of basis matrix  $g \in GL(d)$  taking one to the other. Similarly, multiplying the matrix on the left by any invertible matrix simply replaces the original basis choice by some new one.

Thus, we can think of the set of  $d$ -dimensional subspaces of  $k^n$  as the orbits of the left action of  $GL(d)$  on the set of full-rank  $d \times n$  matrices.

(d.) Let  $W \subset V$  be a  $d$ -dimensional subspace, and let  $A$  be the matrix corresponding to  $W$  as above. Since  $A$  has full rank, the determinant of some  $d \times d$  minor doesn't vanish. For simplicity, say it is the minor given by columns  $1, \dots, d$  (hereafter referred to as the first minor). We show

there is unique matrix of the following form corresponding to  $W$ :

$$\begin{pmatrix} 1 & \cdots & 0 & a_{1,d+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{d,d+1} & \cdots & a_{d,n} \end{pmatrix}$$

Indeed, let  $g \in GL(d)$  be the inverse matrix to the first minor. Then  $gA$  has this form and also determines  $W$ . Further, if  $B$  is another  $d \times n$  matrix which determines  $W$  and has this form, we must have  $B = hgA$  for some  $h \in GL(d)$ . Since the first minor of  $gA$  is the identity matrix, the first minor of  $hgA$  is  $h$ . Thus, we must have that  $h$  is the identity matrix, so  $B = hgA = gA$ .

Denote the set of subspaces  $W$  such that the determinant of the first minor of a matrix representative for  $W$  doesn't vanish by  $U_{(1,\dots,d)}$ . Then, we see that there is a bijective correspondence between  $U_{(1,\dots,d)}$  and  $\mathbb{A}^{d(n-d)}$ , where points in the affine space correspond to the remaining  $d(n-d)$  entries in the matrix for  $W$  after we've put the matrix for  $W$  in the above form.

Now, fix the following notation. For a  $d \times n$  matrix  $A$  and a collection of indices  $1 \leq i_1 < \cdots < i_d \leq n$ , let  $\Delta_{(i_1,\dots,i_d)}(A)$  denote the determinant of the  $d \times d$  minor of  $A$  given by the columns  $i_1, \dots, i_d$ . For any collection of indices  $1 \leq i_1, \dots, i_d \leq n$ , we can construct the set  $U_{(i_1,\dots,i_d)}$  where  $\Delta_{(i_1,\dots,i_d)}$  does not vanish. This set corresponds to  $\mathbb{A}^{d(n-d)}$  in exactly the same way as above. Since there are  $\binom{n}{d}$  such collections of indices, we conclude that we can cover  $\mathbb{G}_d(V)$  by  $\binom{n}{d}$  copies of  $\mathbb{A}^{d(n-d)}$ .

This really is analogous to the standard affine cover of  $\mathbb{P}(V)$ . Considering  $\mathbb{P}(V) = \mathbb{G}_1(V)$ , we're interested in  $1 \times n$  matrices modulo the action of  $1 \times 1$  invertible matrices, i.e. non-zero scalars. The standard affine charts can be described as the non-vanishing of the  $j$ -th homogeneous coordinate, which is precisely the determinant of the  $j$ -th  $1 \times 1$  minor of the  $1 \times n$  matrix.

(e.) For simplicity, we will assume  $U_2 = U_{(1,\dots,d)}$  and  $U_1 = U_{(i_1,\dots,i_d)}$  is any of the other patches. Let  $W$  be a  $d$ -dimensional subspace of  $V$  which is in  $U_1$ , and let  $B$  be the matrix corresponding to  $W$  in the appropriate form. Then  $W$  is in  $U_2$  if and only if  $\Delta_{(1,\dots,d)}(B) \neq 0$ . The map  $\phi_1$  identifies the coordinate functions on  $\mathbb{A}^{d(n-d)}$  with the entries of  $B$ . Since  $\Delta_{(1,\dots,d)}$  is a polynomial in the entries of  $B$ ,  $\Delta_{(1,\dots,d)} \circ (\phi_1)^{-1}$  is polynomial on  $\mathbb{A}^{d(n-d)}$ . Thus,  $V_1$  is open in  $\mathbb{A}^{d(n-d)}$ .

Let's assume  $W \in U_1 \cap U_2$ . Let  $A$  be the matrix corresponding to  $W$  in the form appropriate for  $U_2$ . Then we saw above that  $A = gB$ , where  $g \in GL(d)$  was the inverse matrix to the first minor of  $B$ . Write

$$A = \begin{pmatrix} 1 & \cdots & 0 & a_{1,d+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{d,d+1} & \cdots & a_{d,n} \end{pmatrix}.$$

Note that  $a_{i,j} = \Delta_{(1,\dots,\hat{i},\dots,d,j)}(A)$  for  $j = d+1, \dots, n$  and  $i = 1, \dots, d$ . Thus, we have

$$a_{i,j} = \Delta_{(1,\dots,\hat{i},\dots,d,j)}(gB) = \det(g) \cdot \Delta_{(1,\dots,\hat{i},\dots,d,j)}(B) = \frac{\Delta_{(1,\dots,\hat{i},\dots,d,j)}(B)}{\Delta_{(1,\dots,d)}(B)}.$$

The map  $\phi_2$  identifies the coordinate functions of  $\mathbb{A}^{d(n-d)}$  with the  $a_{i,j}$ . After also using  $\phi_1$  to identify the entries of  $B$  with the coordinates of  $\mathbb{A}^{d(n-d)}$ , these are the component functions of the chart change map  $\phi_2 \circ (\phi_1)^{-1} : V_1 \rightarrow V_2$ . In particular, note that they are rational functions on  $\mathbb{A}^{d(n-d)}$  which are regular on  $V_1$ .

(f.) The above calculation allows us to give  $\mathbb{G}_d(V)$  the structure of an abstract variety over any field. We simply use the bijective maps  $\phi_{(i_1, \dots, i_d)} : U_{(i_1, \dots, i_d)} \rightarrow \mathbb{A}^{d(n-d)}$  to give  $\mathbb{G}_d(V)$  its topology and sheaf of rings locally. By the previous part, this is well defined. Essentially, the maps  $\phi_2 \circ (\phi_1)^{-1}$  are telling us how to glue these copies of  $\mathbb{A}^{d(n-d)}$  to get  $\mathbb{G}_d(V)$ . The point is that these coordinate change maps are regular. If we are working over  $\mathbb{R}$ , noting that the coordinate change maps are smooth shows that  $\mathbb{G}_d(V)$  is a smooth manifold. Since they are also holomorphic when considered over  $\mathbb{C}$ , we can consider  $\mathbb{G}_d(V)$  as a complex manifold. No matter what we do, however, the dimension is  $d(n-d)$  – it locally looks like  $\mathbb{A}^{d(n-d)}$ !