MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 5 SOLUTIONS

Problem 1. (a.) Let $x_0, \dots, x_n$ denote the homogeneous coordinates of $\mathbb{P}^n$, $y_0, \dots, y_m$ the homogeneous coordinates of $\mathbb{P}^m$, and z_{ij} for $i = 0, \dots, n$ and $j = 0, \dots, m$ the homogeneous coordinates of $\mathbb{P}^{(n+1)(m+1)-1}$. If $X = \Sigma_{n,m} \cap H$, where $H = \mathbb{V}(\sum c_{ij} z_{ij})$ is a hyperplane, then the corresponding closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ is defined by the bihomogeneous polynomial $\sum c_{ij} x_i y_j$.

(b.) Let $f(x, y, u, v) = x^2 y^3 uv + x^5 v^2 + y^4 x u^2$. Then $X = \mathbb{V}(f) = \mathbb{V}(u^3 f, v^3 f)$, and

$$\begin{aligned}\sigma_{1,1}(X) &= \Sigma_{1,1} \cap \mathbb{V}(z_{00}^2 z_{10}^2 z_{11} + z_{00}^3 z_{01}^2 + z_{00} z_{10}^4, z_{00} z_{01} z_{11}^3 + z_{01}^5 + z_{00} z_{10} z_{11}^3) \\ &= \mathbb{V}(z_{00} z_{11} - z_{10} z_{01}, z_{00}^2 z_{10}^2 z_{11} + z_{00}^3 z_{01}^2 + z_{00} z_{10}^4, z_{00} z_{01} z_{11}^3 + z_{01}^5 + z_{00} z_{10} z_{11}^3).\end{aligned}$$

Problem 2. (a.) Suppose F and G have a common factor H of degree $d \geq 1$. Then $U^{d-1} FG/H$ has degree $m + n - 1$ and is divisible by F and G. Conversely, suppose F and G divide L of degree $m + n - 1$. Then G divides $L = F(L/F)$, and since L/F has degree $n - 1$, $G \nmid L/F$. Thus G has a common factor with F.

(b.) The first n of these form a basis for for V_F, and the next m form a basis for V_G. To say these are linearly dependent is precisely to say V_F and V_G intersect nontrivially, and the conclusion follows from (a.).

(c.) Write $F = a_0 U^m + a_1 U^{m-1} V + \cdots + a_m V^m$ and $G = b_0 U^n + b_1 U^{n-1} V + \cdots + b_n V^n$. We have

$$\begin{aligned}U^{n-1-j} V^j F &= a_0 U^{m+n-1-j} V^j + a_1 U^{m+n-2-j} V^{j+1} + \cdots + a_m U^{n-j-1} V^{m+j} \\ U^{m-1-k} V^k G &= b_0 U^{m+n-1-k} V^k + b_1 U^{m+n-2-k} V^{k+1} + \cdots + b_n U^{m-k-1} V^{n+k}\end{aligned}$$

for $0 \leq j \leq n - 1$ and $0 \leq k \leq m - 1$. Thus, with respect to the ordered basis

$$U^{m+n-1}, U^{m+n-2} V, \dots, U^{m+n-1-i} V^i, \dots, V^{m+n-1}$$

of $\mathrm{Sym}^{m+n-1}(k^2)^*$, we can represent these vectors by the rows of the matrix in Figure 1., where a blank entry should be considered as zero. Thus, the polynomials in (b.) are linearly dependent if and only if the determinant of this matrix vanishes, which we have already seen is equivalent to F and G having a common factor.

(d.) First of all, note that Γ is certainly nonempty; for example, $(U^m, U^n) \in \Gamma$ since all powers of U vanish at $[0 : 1]$. Further, it is easy to see that two polynomials $F \in \mathrm{Sym}^m(k^2)^*$ and $G \in \mathrm{Sym}^n(k^2)^*$ satisfy $\mathbb{V}(F) \cap \mathbb{V}(G) \neq \emptyset$ in $\mathbb{P}^1$ if and only if they have a common factor. We see from the previous part that the resultant of F and G is a bihomogeneous polynomial in the coefficients of F and G, of bidegree (n, m), respectively. This follows immediately from the fact that the first n rows of the

FIGURE 1.

$$\begin{pmatrix}
a_0 & a_1 & \cdots & a_m & & & & & \\
 & a_0 & a_1 & \cdots & a_m & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & a_0 & a_1 & \cdots & a_m \\
b_0 & b_1 & \cdots & \cdots & b_n & & & & \\
 & b_0 & b_1 & \cdots & \cdots & b_n & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & b_0 & b_1 & \cdots & \cdots & b_n
\end{pmatrix}$$

matrix in Figure 1. use the coefficients of F, while the remaining m rows use the coefficients of G. Thus, Γ is a closed subset of $\mathbb{P}(\mathrm{Sym}^m(k^2)^*) \times \mathbb{P}(\mathrm{Sym}^n(k^2)^*)$.

Problem 3. (a.) The point $p = (a, b) \in A^2$ is on the line $\ell = [c : d] \in \mathbb{P}^1$ if and only if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has linearly dependent rows, i.e. $ad - bc = 0$. Let x, y be the (affine) coordinates on $\mathbb{A}^2$, and s, t the (homogeneous) coordinates on $\mathbb{P}^1$. Since the polynomial $xt - ys$ is homogeneous in s and t, we have that $X = \mathbb{V}(xt - ys)$ is closed in $\mathbb{A}^2 \times \mathbb{P}^1$.

(b.) Suppose first that $p = (a, b) \neq (0, 0)$. Then $\pi^{-1}(\{p\}) = \mathbb{V}(x - a, y - b, xt - ys) = \mathbb{V}(x - a, y - b, at - bs)$ is the single point $\{((a, b), [a : b])\}$, and has dimension zero. However, if $p = (0, 0)$, then $\pi^{-1}(\{p\}) = \mathbb{V}(x, y, xt - ys) = \mathbb{V}(x, y) = \{(0, 0)\} \times \mathbb{P}^1 \simeq \mathbb{P}^1$ has dimension one. Certainly π is not finite, since the fiber over $(0, 0)$ is not a finite set.

(c.) The map η is certainly regular, since it is the restriction to X of the second projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ which is regular. If $\ell = [c : d] \in \mathbb{P}^1$, then $((c, d), [c : d]) \in \eta^{-1}(\{\ell\})$, so η is surjective. In fact, we have that $\eta^{-1}(\ell) = \mathbb{V}(dx - cy, ds - ct)$ is precisely the line $dx - cy = 0$ in $\mathbb{A}^2$ (which is ℓ) cross the point $\ell = [c : d]$ in $\mathbb{P}^1$, and thus $\eta^{-1}(\ell)$ can be identified with the line ℓ itself (hence the name "tautological line bundle." Again, since η does not have finite fibers, it is not finite.

Problem 4. (a.) Consider the map $\mathscr{M} : \mathbb{P}(V^*) \times \mathbb{P}(V^*) \to \mathbb{P}(\mathrm{Sym}^2(V^*))$ which maps a pair of linear forms to the quadratic form given by their product. Explicitly, choosing homogeneous coordinates x, y, z on $\mathbb{P}(V)$, we have

$$(a_0x + a_1y + a_2z) \cdot (b_0x + b_1y + b_2z)$$
$$=$$
$$(a_0b_0)x^2 + (a_1b_1)y^2 + (a_2b_2)z^2 + (a_0b_1 + a_1b_0)xy + (a_0b_2 + a_2b_0)xz + (a_1b_2 + a_2b_1)yz.$$

Thus, under the corresponding identifications of $\mathbb{P}(V^*)$ with $\mathbb{P}^2$ and $\mathbb{P}(\mathrm{Sym}^2(V^*))$ with $\mathbb{P}^5$, we have

$$\begin{array}{rccc} \mathscr{M}: & \mathbb{P}^2 \times \mathbb{P}^2 & \to & \mathbb{P}^5 \\ & ([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) & \mapsto & [a_0b_0 : a_1b_1 : a_2b_2 : a_0b_1 + a_1b_0 : a_0b_2 + a_2b_0 : a_1b_2 + a_2b_1]. \end{array}$$

The image of $\mathscr{M}$ is precisely the set of degenerate conics. In particular, it is a closed subvariety of $\mathbb{P}^5$. Since there are many non-degenerate conics, e.g. $x^2 - yz$, it is a proper subvariety.

Consider the Segre embedding $\sigma_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$, which is given by $\sigma_{2,2}(([a_i], [b_j])) = [a_ib_j]$. If c_{ij}, $0 \le i, j \le 2$, are the coordinates on $\mathbb{P}^8$, we see immediately that $\mathscr{M}$ factors as

$$\begin{array}{ccc} \mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{\mathscr{M}} & \mathbb{P}^5 \\ & {\scriptstyle \sigma_{2,2}} \searrow \quad \nearrow {\scriptstyle \pi} & \\ & \mathbb{P}^8 & \end{array}$$

where $\pi : \mathbb{P}^8 \dashrightarrow \mathbb{P}^5$ is the linear projection given by

$$\begin{bmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{bmatrix} \mapsto [c_{00} : c_{11} : c_{22} : c_{10} + c_{01} : c_{20} + c_{02} : c_{12} + c_{21}].$$

Since $\sigma_{2,2}$ is an isomorphism onto its image $\Sigma_{2,2}$, it follows that π is regular on $\Sigma_{2,2}$, and the image of $\mathscr{M}$ equals $\pi(\Sigma_{2,2})$. Further, since π is a linear projection defined on $\Sigma_{2,2}$, it induces a finite map onto its image. Since finite maps preserve dimension, we conclude that the dimension of the subvariety of degenerate conics is 4.

(b.) Consider the map $\mathscr{S} : \mathbb{P}(V^*) \to \mathbb{P}(\mathrm{Sym}^2(V^*))$ which takes a linear form to its square. We have

$$(a_0x + a_1y + a_2z)^2 = ({a_0}^2)x^2 + ({a_1}^2)y^2 + ({a_2}^2)z^2 + (2a_0a_1)xy + (2a_0a_2)xz + (2a_1a_2)yz.$$

Thus, under the corresponding identifications of $\mathbb{P}(V^*)$ with $\mathbb{P}^2$ and $\mathbb{P}(\mathrm{Sym}^2(V^*))$ with $\mathbb{P}^5$, we have

$$\begin{array}{rccc} \mathscr{S}: & \mathbb{P}^2 & \to & \mathbb{P}^5 \\ & [a_0 : a_1 : a_2] & \mapsto & [{a_0}^2 : {a_1}^2 : {a_2}^2 : 2a_0a_1 : 2a_0a_2 : 2a_1a_2]. \end{array}$$

The image of $\mathscr{S}$ is precisely the set of “double lines.” In particular, it is a proper ($x^2 - yz$ is not a “double line”) closed subset of the space of all conics, i.e. $\mathbb{P}^5$.

Consider the linear automorphism $\varphi : \mathbb{P}^5 \to \mathbb{P}^5$ given by

$$[d_0 : d_1 : d_2 : d_3 : d_4 : d_5] \mapsto [d_0 : d_1 : d_2 : \frac{1}{2}d_3 : \frac{1}{2}d_4 : \frac{1}{2}d_5].$$

The composition $\varphi \circ \mathscr{S}$ is precisely the Veronese embedding

$$\begin{array}{rccc} \nu_2: & \mathbb{P}^2 & \to & \mathbb{P}^5 \\ & [a_0 : a_1 : a_2] & \mapsto & [{a_0}^2 : {a_1}^2 : {a_2}^2 : a_0a_1 : a_0a_2 : a_1a_2]. \end{array}$$

Thus, φ induces an isomorphism between the subset of "double lines" and the Veronese surface.

Problem 5. (a.) Let $f(x_0, \dots, x_n)$ be the homogeneous polynomial defining V. Then, for any $y = [y_0 : \cdots : y_{n-1}] \in \mathbb{P}^{n-1}$, we have

$$\pi^{-1}(\{y\}) = \{[y_0 : \cdots : y_{n-1} : z] \in \mathbb{P}^n \quad | \quad f(y_0, \dots, y_{n-1}, z) = 0\}.$$

In particular, we see that $\pi^{-1}(\{y\})$ is naturally identified with the set of roots of the one-variable polynomial

$$f(y_0, \dots, y_{n-1}, T) = T^d + a_1(y_0, \dots, y_{n-1})T^{d-1} + \cdots + a_d(y_0, \dots, y_{n-1}).$$

Since we are working over an algebraically closed field and $f(y_0, \dots, y_{n-1}, T)$ is a monic polynomial of degree d, we know that $f(y_0, \dots, y_{n-1}, T)$ has exactly d roots when counted with multiplicity. This allows us to assign a multiplicity to the points in the preimage of $\{y\}$; it is simply the multiplicity of the corresponding root of $f(y_0, \dots, y_{n-1}, T)$. Thus, we see π has degree d. Since $d > 0$, π is surjective.

(b.) From above, it is clear that $y = [y_0, \dots, y_{n-1}] \in \mathbb{P}^{n-1}$ is a ramification point if and only if $f(y_0, \dots, y_{n-1}, T)$ has a repeated root. From elementary Galois theory, we know this happens if and only if $f(y_0, \dots, y_{n-1}, T)$ and its formal derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(y_0, \dots, y_{n-1}, T)) = dT^{d-1} + (d-1)a_1(y_0, \dots, y_{n-1})T^{d-2} + \cdots + a_{d-1}(y_0, \dots, y_{n-1})$$

have a root in common. The general idea, which will be made formal below, is that this can be checked using the resultant of these two polynomials. Note that $\frac{\mathrm{d}}{\mathrm{d}t}(f(y_0, \dots, y_{n-1}, T))$ is not the zero polynomial because we are working over a field of characteristic zero.

First, consider a nonzero polynomial g on $\mathbb{A}^1$. Let G be its homogenizations to $\mathbb{P}^1$. We leave it to the reader to check that the roots of G in $\mathbb{P}^1$ are precisely the roots of g in $\mathbb{A}^1 \subseteq \mathbb{P}^1$. Thus, if h is another polynomial with homogenization H, we see using Problem 2 that g and h have a common root in $\mathbb{A}^1$ if and only if the resultant $\mathrm{Res}(G, H)$ is nonzero. Note that the homogenization procedure equates the coefficients of g and G, as well as those of h and H. Thus, $\mathrm{Res}(G, H)$ is just the determinant of a certain polynomial whose entries are the coefficients of g and h. Partly for this reason, we will denote $\mathrm{Res}(G, H)$ by $\mathrm{Res}(g, h)$.

So we have the $[y_0, \dots, y_{n-1}] \in \mathbb{P}^{n-1}$ is a ramification point if and only if

$$R(y_0, \dots, y_{n-1}) := \mathrm{Res}\left(f(y_0, \dots, y_{n-1}, T), \frac{\mathrm{d}}{\mathrm{d}t}(f(y_0, \dots, y_{n-1}, T))\right)$$

is zero. It is clear from the construction of the resultant that R is a polynomial function in $y_0, \dots, y_{d-1}$. However, it is not immediately obvious whether it is homogeneous or not. Certainly, if $\lambda \neq 0$, we have

$$R(y_0, \dots, y_{n-1}) = 0 \quad \text{if and only if} \quad R(\lambda y_0, \dots, \lambda y_{n-1}) = 0$$

since $[y_0 : \cdots : y_{n-1}] = [\lambda y_0 : \cdots : \lambda y_{n-1}]$ in $\mathbb{P}^{n-1}$. In particular, the ideal $\sqrt{\langle R(y_0, \dots, y_{n-1})\rangle}$ inside of $k[y_0, \dots, y_{n-1}]$ is invariant under the action of k^*. Thus, $\sqrt{\langle R(y_0, \dots, y_{n-1})\rangle}$ is a homogeneous ideal. Since $\sqrt{\langle R(y_0, \dots, y_{n-1})\rangle}$ is principal, it is easy to see that $R(y_0, \dots, y_{n-1})$ is homogeneous.

Thus, we conclude that the ramification locus in $\mathbb{P}^{n-1}$ is a closed set defined by the vanishing of a single homogeneous polynomial $R(y_0, \dots, y_{n-1})$. However, it is not clear that $R(y_0, \dots, y_{n-1})$ is not the zero polynomial, so it remains to show that the ramification locus is a proper subset of $\mathbb{P}^{n-1}$. We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(y_0, \dots, y_{n-1}, T)) = \frac{\partial f}{\partial x_n}(y_0, \dots, y_{n-1}, T),$$

so that the ramification locus is precisely $\pi(\mathbb{V}(f, \frac{\partial f}{\partial x_n}))$. Since we are working over a field of characteristic zero, $\frac{\partial f}{\partial x_n}$ is a nonzero homogeneous polynomial of degree $d-1$. In particular, $\frac{\partial f}{\partial x_n}$ is not divisible by f, and $\mathbb{V}(f, \frac{\partial f}{\partial x_n})$ is a proper subset of $V = \mathbb{V}(f)$. Thus, the dimension of (any irreducible component of) $\mathbb{V}(f, \frac{\partial f}{\partial x_n})$ is $n-2$. It follows that the ramification locus has dimension at most $n-2$, and cannot equal all of $\mathbb{P}^{n-1}$

(c.) Consider the affine open cover $U_i = \mathbb{P}^{n-1} \setminus \mathbb{V}(y_i)$ for $0 \le i \le n-1$. The coordinate ring is

$$k[U_i] = k[\frac{y_0}{y_i}, \dots, \frac{y_{n-1}}{y_i}] \simeq k[\frac{x_0}{x_i}, \dots, \frac{x_{n-1}}{x_i}].$$

We have that $V_i := \pi^{-1}(U_i) = V \setminus \mathbb{V}(x_i)$ is also affine with coordinate ring

$$k[V_i] = k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]/\langle f(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i})\rangle = k[U_i][\frac{x_n}{x_i}]/\langle f(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i})\rangle,$$

and the homomorphism induced by π is simply the inclusion. Since we have that

$$f(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}) = (\frac{x_n}{x_i})^d + a_1(\frac{x_0}{x_i}, \dots, \frac{x_{n-1}}{x_i})(\frac{x_n}{x_i})^{d-1} + \cdots + a_d(\frac{x_0}{x_i}, \dots, \frac{x_{n-1}}{x_i})$$

is a monic polynomial in $\frac{x_n}{x_i}$ with coefficients in $k[U_i]$, this is an integral extension of rings. Thus, π is a finite map.

Problem 6. (a.) If W is a subspace of dimension d, consider two bases $e_1, \dots, e_d$ and $f_1, \dots, f_d$. We have

$$f_1 \wedge \cdots \wedge f_d = \det(g) e_1 \wedge \cdots \wedge e_d$$

where $g \in \mathrm{GL}(W)$ is the linear transformation sending e_i to f_i. In particular, $\bigwedge W$ is the one-dimensional subspace of $\bigwedge V$ spanned by $e_1 \wedge \cdots \wedge e_d$ for any basis $e_1, \dots, e_d$ of W, and ϕ is well-defined. Its image certainly consists of one-dimensional subspaces of $\mathbb{P}(\bigwedge^d V)$ spanned by indecomposable vectors. Conversely, if $v_1 \wedge \cdots \wedge v_d$ is a nonzero indecomposable vector, then $v_1, \dots, v_d$ must be linearly independent. Thus, $v_1, \dots, v_d$ are a basis for a d-dimensional subspace, and the linear span of $v_1 \wedge \cdots \wedge v_d$ is in the image of ϕ.

If W_1 and W_2 are distinct d-dimensional subspaces, then we can find a basis $e_1, \dots, e_n$ for V such that $e_1, \dots, e_d$ are a basis for W_1 and $e_1, \dots, e_k, e_{d+1}, \dots, e_{2d-k}$ are a basis for W_2. Recall that

$\{e_{i_1} \wedge \cdots \wedge e_{i_d}\}_{1 \le i_1 < \cdots < i_d \le n}$ form a basis for $\bigwedge^d V$. In particular,

$$e_1 \wedge \cdot \wedge e_d \quad \text{and} \quad e_1 \wedge \cdots e_k \wedge e_{d+1} \wedge \cdots e_{2d-k}$$

are linearly independent and span distinct one-dimensional subspaces of $\bigwedge^d V$. Thus,

$$\phi(W_1) = [e_1 \wedge \cdot \wedge e_d] \neq [e_1 \wedge \cdots e_k \wedge e_{d+1} \wedge \cdots e_{2d-k}] = \phi(W_2),$$

and ϕ is injective onto its image.

(b.) Fix a basis $e_1, \ldots, e_n$ of V, and let $f_1, \ldots, f_d$ be a basis for a d-dimensional subspace W. If we write $f_i = \sum_{i=1}^n a_{ij} e_j$, then W is identified with the row-space of the $d \times n$ matrix $A = (a_{ij})$. For $1 \le i_1 < \cdots < i_d \le n$, we write $\Delta_{i_1,\ldots,i_d}(A)$ for the determinant of the maximal minor of A consisting of the columns $i_1, \ldots, i_d$. By elementary linear algebra, we have

$$f_1 \wedge \cdots \wedge f_d = \sum_{1 \le i_1 < \cdots < i_d \le n} \Delta_{i_1,\ldots,i_d}(A)\, e_{i_1} \wedge \cdots \wedge e_{i_d}.$$

Since $\{e_{i_1} \wedge \cdots \wedge e_{i_d}\}_{1 \le i_1 < \cdots < i_d \le n}$ form a basis for $\bigwedge^d V$, it follows that ϕ can be expressed in coordinates as taking W to the N-tuple consisting of the determinants of the maximal minors of a matrix representing W. Note that any other $d \times n$ matrix B representing W can be written as gA for some $g \in \mathrm{GL}(d)$. Since we have $\Delta_{i_1,\ldots,i_d}(B) = \det(g)\Delta_{i_1,\ldots,i_d}(A)$, we see once more that this is well-defined.

(c.) Note that if $\omega = v \wedge \eta$ for some $\eta \in \bigwedge^{d-1} V$, then $v \in \ker(\wedge\omega)$. Conversely, suppose that $v_1, \ldots, v_k$ are linearly independent vectors in $\ker(\wedge\omega)$. We will show that $k \le d$ and $\omega = v_1 \wedge \cdots v_k \wedge \omega'$ for some $\omega' \in \wedge^{d-k} V$. Choose a basis $e_1, \ldots, e_n$ of V with $v_j = e_j$ for $1 \le j \le k$. Write

$$\omega = \sum_{1 \le i_1 < \cdots < i_d \le n} a_{i_1,\ldots,i_d} e_{i_1} \wedge \cdots \wedge e_{i_d}.$$

Since $v_j \wedge \omega = 0$ and $\{e_{i_1} \wedge \cdots \wedge e_{i_{d+1}}\}_{1 \le i_1 < \cdots < i_{d+1} \le n}$ form a basis for $\bigwedge^{d+1} V$, we must have $a_{i_1,\ldots,i_d} = 0$ whenever $j \notin \{i_1, \ldots, i_d\}$. Since $\omega \neq 0$, it follows that $k \le d$, and we have

$$\omega = v_1 \wedge \cdots \wedge v_k \wedge \Big(\sum_{k+1 \le i_{k+1} < \cdots < i_d \le n} a_{1,\ldots,k,i_{k+1},\ldots,i_d} e_{i_{k+1}} \wedge \cdots \wedge e_{i_d} \Big).$$

It follows that $\dim(\ker(\wedge\omega)) \le d$, with equality if and only if ω is primitive. Equivalently, $\mathrm{rank}(\wedge\omega) \ge n - d$ with equality if and only if ω is primitive.

(d.) We have a linear transformation

$$\begin{array}{ccc} \bigwedge^d V & \to & \mathrm{Hom}(V, \bigwedge^{d+1} V) \\ \omega & \mapsto & \wedge\omega. \end{array}$$

Since the closed subset of $\text{Hom}(V, \bigwedge^{d+1} V)$ consisting of transformations of rank $\leq n-d$ is defined by homogeneous forms (i.e. the vanishing of the determinants of the $(n-d+1)\times(n-d+1)$ minors of a matrix representation) on $\text{Hom}(V, \bigwedge^{d+1} V)$, it follows that the collection of $\omega \in \bigwedge^d V$ with $\text{rank}(\wedge\omega) \leq n-d$ is also defined by homogeneous forms. Thus, using (c.), we see that ϕ identifies $\mathbb{G}(d,n)$ with a closed subvariety of $\mathbb{P}(\bigwedge^d V)$.

(e.) Choosing a basis for V, let $\{\Delta_{i_1,\ldots,i_d}\}_{1\leq i_1<\cdots<i_d\leq n}$ denote the homogeneous coordinates on $\mathbb{P}(\bigwedge^d V)$. Let $U_{i_1,\ldots,i_d} = \mathbb{P}(\bigwedge^d V) \setminus \mathbb{V}(\Delta_{i_1,\ldots,i_d})$ be the corresponding standard affine cover. From part (b.), it follows that $\phi^{-1}(U_{i_1,\ldots,i_d})$ is precisely the collection of d-dimensional subspaces W with a corresponding $d \times n$ matrix A such that $\Delta_{i_1,\ldots,i_d}(A) \neq 0$. In particular, this is an affine open subset of $\mathbb{G}(d,n)$ described in the previous problem set.

Let $\mathscr{M}$ be the set of $d\times n$ matrices with rank d. Since this is an open subset of $\mathbb{A}^{d\times n}$ (where the determinant of at least one maximal minor does not vanish), it is irreducible. There is a surjective morphism $\mathscr{M} \to \mathbb{G}(d,V)$ which maps a matrix A onto its row space W. Since the image of an irreducible set under a continuous map is irreducible, it follows that $\mathbb{G}(d,n)$ is irreducible. Since it is covered by affine patches isomorphic to $\mathbb{A}^{d(n-d)}$, its dimension is $d(n-d)$.