MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 6 SOLUTIONS

Problem 1. (a.) If $X \subseteq \mathbb{P}^n$ is a projective variety, we will write $\mathscr{C}(X)$ for the cone over X in $\mathbb{A}^{n+1}$. The homogeneous radical ideal $\mathbb{I}(X)$ defining X in $\mathbb{P}^n$ is equal to the ideal $\mathbb{I}(\mathscr{C}(X))$ defining $\mathscr{C}(X)$ in $\mathbb{A}^{n+1}$. In particular, X is irreducible if and only if $\mathscr{C}(X)$ is irreducible (in which case $\mathbb{I}(X) = \mathbb{I}(\mathscr{C}(X))$ is a prime ideal). More generally, if $X = X_1 \cup \cdots \cup X_r$ is a decomposition of X into its irreducible components, then $\mathscr{C}(X) = \mathscr{C}(X_1) \cup \cdots \cup \mathscr{C}(X_r)$ is a decomposition of $\mathscr{C}(X)$ into its irreducible components. Thus, in order to calculate $\dim(\mathscr{C}(X))$, we may and will assume X and $\mathscr{C}(X)$ are irreducible.

Consider the natural surjection

$$\rho : \mathscr{C}(X) \setminus \{0\} \to X$$
$$(p_0, \ldots, p_n) \mapsto [p_0 : \cdots : p_n].$$

If $p = [p_0 : \cdots : p_n] \in X$, then $\rho^{-1}(\{p\}) = \{(\lambda p_0, \ldots, \lambda p_n) \mid \lambda \in \mathbb{A}^1 \setminus \{0\}\} \simeq \mathbb{A}^1 \setminus \{0\}$ has dimension one. By the theorem on the dimension of fibers, $\dim(\mathscr{C}(X)) = \dim(X) + 1$.

(b.) Let X be the variety of $m \times n$ matrices of rank t. Consider the open subset U of X where the upper left $t \times t$ minor (i.e. given by the first t rows and first t columns) has nonzero determinant. Such a matrix has the form

$$\left(\begin{array}{c|c} A & B \\ \hline \multicolumn{2}{c}{C} \end{array}\right)$$

where $A \in GL(t)$, B is an arbitrary $t \times (n-t)$ matrix, and C is an $(m-t) \times n$ matrix whose rows are linear combinations of the rows of the matrix

$$\left(\begin{array}{c|c} A & B \end{array}\right).$$

In fact, since the above matrix has full rank, the rows of C are linear combinations of its rows in a unique way. Thus, we can construct a bijective morphism

$$GL(t) \times \mathbb{A}^{t \times (n-t)} \times \mathbb{A}^{(m-t)\times t} \to U$$

which sends a triple (G, N, M) to the matrix

$$\left(\begin{array}{c|c} G & N \\ \hline \multicolumn{2}{c}{M \cdot (G \,|\, N)} \end{array} \right).$$

From the theorem on the dimension of fibers, it follows that U is an irreducible variety of dimension $t^2 + t(n-t) + (m-t)t = t(m+n-t)$. Analogous constructions show the same is true for the open subsets of X where the other $t \times t$ minors have nonzero determinant. Since these open subsets cover X, it follows that X has dimension $t(m+n-t)$.

(c.) The twisted cubic is the image of the Veronese embedding $\nu_3 : \mathbb{P}^1 \to \mathbb{P}^3$, it is isomorphic to $\mathbb{P}^1$ and hence has dimension 1.

(d.) Suppose X is an irreducible variety of dimension d, and X' is an irreducible variety of dimension d'. Then $X \times X'$ is irreducible, and the first projection morphism

$$p_1 : X \times X' \to X$$

is surjective with every fiber isomorphic to X'. By the theorem on the dimension of fibers, it follows that $X \times X'$ has dimension $d + d'$.

Problem 2. (a.) Hypersurfaces of degree d in $\mathbb{P}^n = \mathbb{P}(V)$ are parametrized by $\mathbb{P}(\mathrm{Sym}^d(V^*))$. For any fixed point Q, the collection of hypersurfaces of degree d passing through Q is a hyperplane in $\mathbb{P}(\mathrm{Sym}^d(V^*))$. Furthermore, distinct points correspond to distinct hypersurfaces. Thus, the hypersurfaces of degree d passing through Q but not $P_1, \ldots, P_t$ is a nonempty open subset of the hyperplane of hypersufaces of degree d passing through Q.

(b.) For the moment, let $\mathrm{kdim}(V)$ be the length of the longest chain of irreducible subvarieties of V. Since a proper closed subset always has smaller dimension, it is clear that $\mathrm{kdim}(V) \leq \dim(V)$. We proceed to show the other inequality by induction on the dimension of V. If $\dim(V) = 0$, then V is a point and the statement is clear. Thus, assume $V \subseteq \mathbb{P}^n$ has positive dimension. Choose two distinct points Q and P in V. From above, we know we can find a hyperplane H in $\mathbb{P}^n$ such that $Q \in H$ but $P \notin H$. Thus, $H \cap V$ is a nontrivial hypersurface in V, and so each irreducible component of $H \cap V$ has dimension $\dim(V) - 1$. Let W be one such irreducible component. By induction, $\mathrm{kdim}(W) = \dim(W)$; also, $\mathrm{kdim}(V) \geq \mathrm{kdim}(W) + 1$ since a maximal chain of irreducible subvarieties of W can be extended by adding V to get a chain of irreducible subvarieties of V. Thus, we have

$$\mathrm{kdim}(V) \leq \dim(V) = \dim(W) + 1 = \mathrm{kdim}(W) + 1 \leq \mathrm{kdim}(V)$$

which shows $\text{kdim}(V) = \dim(V)$ and completes the induction.

(c.) Again, it is clear from above that $\dim_x(V) \le \dim(V)$. We will show the opposite inequality by induction on the dimension of V. If $V = \{x\}$, then equality clearly holds. Thus, assume $V \subseteq \mathbb{P}^n$ has positive dimension, and choose another point $x \neq p \in V$. Again, we can find a hyperplane H in $\mathbb{P}^n$ such that $x \in H$ but $p \notin H$. Let W be an irreducible component of $H \cap V$ containing x. By induction, we have $\dim_x(W) = \dim(W) = \dim(V) - 1$. Also, extending a maximal chain of irreducible subvarieties of W starting at $\{x\}$ by adjoining V, we have $\dim_x(W) + 1 \le \dim_x(V)$. Thus, we have

$$\dim_x(V) \le \dim(V) = \dim(W) + 1 = \dim_x(W) + 1 \le \dim_x(V)$$

which completes the induction. It follows that $\dim_x(V) = \dim(V)$ for all $x \in V$.

(d.) Let $x \neq p \in V$, and let H_1 be an arbitrary hyperplane with $x \in H_1$ and $p \notin H_1$. Proceeding inductively, suppose $H_1, \ldots, H_r$ have been chosen $(r < d)$ such that $x \in V \cap H_1 \cap \cdots \cap H_r$ and the dimension of any irreducible component of $V \cap H_1 \cap \cdots \cap H_r$ is $d - r$. Let $W_1, \ldots, W_s$ be the irreducible components of $V \cap H_1 \cap \cdots \cap H_r$. Choose points $x \neq p_i \in W_i$ for $i = 1, \ldots, s$, and let H_{r+1} be a hyperplane in $\mathbb{P}^n$ with $x \in H_{r+1}$ but $p_i \notin H_{r+1}$ for $i = 1, \ldots, s$. Then again we have that $x \in V \cap H_1 \cap \cdots \cap H_{r+1}$ and the dimension of any irreducible component of $V \cap H_1 \cap \cdots \cap H_{r+1}$ is $d - r - 1$. We end with hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^n$ such that $x \in V \cap H_1 \cap \cdots \cap H_d$ and $V \cap H_1 \cap \cdots \cap H_d$ is a finite set of points. Let U be an affine neighborhood of x in V such that $U \cap H_1 \cap \cdots \cap H_d = \{x\}$. Let $f_1, \ldots, f_d$ be equations defining $H_1, \ldots, H_d$ on U, respectively. By construction, the common zero set of $f_1, \ldots, f_d$ on U is precisely $\{x\}$.

Problem 3. (a.) The identification of $k(Y)$ with a subfield of $k(X)$ comes via pullback of rational functions along ϕ. More explicity, take an affine open subset V of Y. Since ϕ is finite, $U := \phi^{-1}(V)$ is also affine and the map induced on coordinate rings by pulling back along ϕ is an integral extension of domains. In particular, it identifies $\mathcal{O}_Y(V)$ with a subring of $\mathcal{O}_X(U)$. Since this is, in fact, a module finite extension of domains, there corresponding inclusion of fraction fields $k(Y) \subseteq k(X)$ is a finite field extension.

(b.) We will use the notation from Problem 5 on Problem set 5. We have that

$$k(\mathbb{P}^{n-1}) = k(\frac{x_1}{x_0}, \ldots, \frac{x_{n-1}}{x_0})$$

and

$$\begin{aligned} k(V) &= \text{Frac}\left(k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] / \langle \frac{x_n}{x_0}^d + a_1(\frac{x_1}{x_0}, \ldots, \frac{x_{n-1}}{x_0}) \frac{x_n}{x_0}^{d-1} + \cdots + a_d(\frac{x_1}{x_0}, \ldots, \frac{x_{n-1}}{x_0}) \rangle \right) \\ &= k(\mathbb{P}^{n-1})[\frac{x_n}{x_0}] / \langle \frac{x_n}{x_0}^d + a_1(\frac{x_1}{x_0}, \ldots, \frac{x_{n-1}}{x_0}) \frac{x_n}{x_0}^{d-1} + \cdots + a_d(\frac{x_1}{x_0}, \ldots, \frac{x_{n-1}}{x_0}) \rangle. \end{aligned}$$

Since $T^d + a_1(\frac{x_1}{x_0}, \dots, \frac{x_{n-1}}{x_0})T^{d-1} + \cdots + a_d(\frac{x_1}{x_0}, \dots, \frac{x_{n-1}}{x_0})$ is an irreducible polynomial in T with coefficients in $k(\mathbb{P}^{n-1})$, this is a finite field extension of degree d. Thus, the degre of π is d.

(c.) We can identify $k(\mathbb{P}^1)$ with $k(T)$ for an indeterminate T. The corresponding field extension is $k(T^d) \subset k(T)$, and has degree d. If $0, \infty \neq [a : b] \in \mathbb{P}^1$, then the fiber over $[a : b]$ is

$$\{[1 : c] \in \mathbb{P}^1 \mid c^d = \frac{b}{a}\}$$

and corresponds to the d-th roots of $\frac{b}{a}$. In general, this depends on $\frac{b}{a}$ and the characteristic of k! If the characteristic of k is zero, these fibers will always have cardinality d. In positive characteristic, however, there will be fibers with less than d points whenever the characteristic divides d. Lastly, note that the fibers over 0 and ∞ are simply 0 and ∞, respectively, regardless of the characteristic.

Problem 4. (a.) Let F be the degree d irreducible homogeneous polynomial defining X. Then $L \cap X$ is defined in $L \simeq \mathbb{P}^1$ by the vanishing of the homogeneous polynomial $F|_L$. This polynomial is identically zero only when $L \subseteq X$, and otherwise it has degree d. Thus, if we count its roots with multiplicity, we see that $L \cap X$ consists of d points whenever $L \not\subset X$.

(b.) Since $X \neq \mathbb{P}^n$, we can choose coordinates such that $[0 : \cdots : 0 : 1] \notin X$. This means, up to multiplication by a scalar, we have

$$F(x_0, \dots, x_n) = x_n^d + a_1 x_n^{d-1} + \cdots + a_d$$

where a_i is a homogeneous polynomial of degree i in $x_0, \dots, x_{n-1}$. We have already seen above that projection from $[0 : \cdots : 0 : 1] \notin X$ to $\mathbb{P}^{n-1}$ is a finite cover of degree d.

Problem 5. (a.) Let $\nu_d : \mathbb{P}^n \to \mathbb{P}(\mathrm{Sym}^d(V^*)) \simeq \mathbb{P}^N$ be the d-th Veronese embedding. We can choose coordinates $z_0, \dots, z_N$ on $\mathbb{P}^N$ such that $z_i \circ \nu_d = F_i$ for $i = 1, \dots, m$. The morphism $\phi : V \to \mathbb{P}^m$ factors as

$$V \xrightarrow{\iota} \mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^N \overset{\pi}{\dashrightarrow} \mathbb{P}^m$$

where ι is the inclusion map, and $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^m$ is the linear projection given by

$$[z_0 : \cdots z_N] \mapsto [z_0 : \cdots : z_m].$$

Since ι and ν_d are closed embeddings, it follows that ϕ is a finite morphism onto its image.

(b.) Fix a point $p \in \mathbb{P}^m$ in the image of ϕ. There is a linear automorphism η of $\mathbb{P}^m$ which sends p to $[1 : 0 : \cdots : 0]$. Like ϕ, the composition $\eta \circ \phi$ can be written as

$$x \in V \longmapsto [G_0(x) : \cdots : G_m(x)]$$

where $G_0, \dots, G_m$ are homogeneous polynomials on $\mathbb{P}^n$ of degree d which do not vanish simultaneously at any point of V. Thus, we have

$$\phi^{-1}(\{p\}) = (\eta \circ \phi)^{-1}([1 : 0 : \cdots : 0]) = \mathbb{V}(G_1, \dots, G_m) \cap V \subseteq \mathbb{P}^n.$$

Assume, by way of contradiction, that $\dim(\phi^{-1}(\{p\})) \geq 1$. Then we know that

$$\dim\left(\phi^{-1}(\{p\}) \cap \mathbb{V}(G_0)\right) \geq 0.$$

In particular, we have

$$\emptyset \neq \phi^{-1}(\{p\}) \cap \mathbb{V}(G_0) = \mathbb{V}(G_0, \dots, G_m) \cap V,$$

so that $G_0, \dots, G_m$ vanish simultaneously at some point of V. This is absurd, hence we must have $\dim(\phi^{-1}(\{p\})) = 0$, i.e. $\phi^{-1}(\{p\})$ is a finite set of points. Since V is a projective variety, and the fibers of ϕ are finite, we have that ϕ is a finite morphism onto its image.

(c.) In general, for fixed n and d but arbitrary V, the fibers can have any number of points in general. However, if $V = \mathbb{P}^n$, one can show that we must have $m = n$ and the cardinality of the fibers is bounded by d^n.

Problem 6. (a.) We have that a point $([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) \in \mathbb{P}^n \times \mathbb{P}^n$ is in $\Delta(\mathbb{P}^n)$ if and only if $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$, which is equivalent to saying the martix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

has rank one. Thus,

$$\Delta(\mathbb{P}^n) = \mathbb{V}(\langle x_i y_j - x_j y_i \,|\, 0 \leq i, j \leq n \rangle)$$

is a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$. Since either one of the two projection maps restricted to $\Delta(\mathbb{P}^n)$ give an inverse to Δ, we see that Δ is an isomorphism onto its image.

(b.) If $V \subseteq \mathbb{P}^n$, then $V \times V$ inherits its topology as a subset of $\mathbb{P}^n \times \mathbb{P}^n$. We see $\Delta_V = \Delta(\mathbb{P}^n) \cap (V \times V)$ is closed in $V \times V$ since $\Delta(\mathbb{P}^n)$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.

(c.) If U_1 and U_2 are affine subvarieties of $V \subseteq \mathbb{P}^n$, then Δ induces an isomorphism between $U_1 \cap U_2$ and $\Delta_V \cap (U_1 \times U_2)$. Since $U_1 \times U_2$ is affine and Δ_V is closed in $V \times V$, it follows that $U_1 \cap U_2$ is affine.

Problem 7. (a.) The proof is practically identical to showing that the Zariski topology on an affine variety is well defined. One checks

$$\bigcap_\alpha \mathbb{V}(I_\alpha) = \mathbb{V}(\bigcap_\alpha I_\alpha) \quad \text{and} \quad \mathbb{V}(I_1) \cup \cdots \cup \mathbb{V}(I_m) = \mathbb{V}(I_1 + \cdots + I_m)$$

so that arbitrary intersections and finite unions of closed sets are closed, and that both

$$\emptyset = \mathbb{V}(R) \quad \text{and } \operatorname{Spec}(R) = \mathbb{V}(\langle 0 \rangle)$$

are closed.

(b.) We first establish a bijection between X and $\mathfrak{m}\mathrm{Spec}(R)$, as follows. If $x \in X$, we let $\mathfrak{m}_x$ be the ideal of functions in $R = \mathcal{O}_X$ vanishing at x. In other words, $\mathfrak{m}_x$ is the kernel of the evaluation morphism $\mathcal{O}_X \to k$ which takes a function f to its value $f(x)$. In particular, $\mathfrak{m}_x$ is a maximal ideal. The map $X \to \mathfrak{m}\mathrm{Spec}(R)$ given by $x \mapsto \mathfrak{m}_x$ is certainly injective: if $x_1 \neq x_2$, there is some coordinate function $f \in \mathcal{O}_x$ with $f(x_1) = 0$ but $f(x_2) \neq 0$, and we have $f \in \mathfrak{m}_{x_1} \setminus \mathfrak{m}_{x_2}$. Conversely, the nullstellensatz guarantees this map is surjective. Suppose $\mathfrak{m} \in \mathfrak{m}\mathrm{Spec}(R)$. Since $\mathfrak{m} \neq R$, we have $\mathbb{V}(\mathfrak{m}) \neq \mathbb{V}(R) = \emptyset$. Choose $x \in \mathbb{V}(\mathfrak{m})$. Then we have $R \neq \mathfrak{m}_x = \mathbb{I}(\{x\}) \supseteq \mathbb{I}(\mathbb{V}(\mathfrak{m})) = \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, we see $\mathfrak{m} = \mathfrak{m}_x$.

To see that this bijection is a homeomorphism, let $Z \subseteq X$ be a closed subset. Then we know $x \in Z$ is equivalent to $\mathfrak{m}_x = \mathbb{I}(\{x\}) \supseteq \mathbb{I}(Z)$. Hence, under the bijection above, Z corresponds to $\mathfrak{m}\mathrm{Spec}(R) \cap \mathbb{V}(\mathbb{I}(Z))$. Since $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ in $\mathrm{Spec}(R)$, and every radical ideal of R has the form $\mathbb{I}(Z)$ for some closed subset $Z \subset X$, it follows that our bijection is an isomorphism.

(c.) To see that the map $\phi_{\mathrm{Spec}} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is well defined, note for any $P \in \mathrm{Spec}(S)$ that ϕ induces an injection

$$R/\phi^{-1}(P) \subseteq S/P$$

so that $\phi^{-1}(P)$ is necessarily a prime ideal. It is continuous since $(\phi_{\mathrm{Spec}})^{-1}(\mathbb{V}(I)) = \mathbb{V}(\phi(I))$.

Suppose R and S are both finitely generated reduced k-algebras with associated affine varieties X and Y, respectively. If $\phi_{\mathrm{Var}} : Y \to X$ is the associated map of affine varieties, we know ϕ is simply given by pulling back regular functions via ϕ_{Var}. Thus, we see for any $y \in Y$ that $\phi^{-1}(\mathfrak{m}_y) = \mathfrak{m}_{\phi_{\mathrm{Var}}(y)}$ is a maximal ideal, and the map of maximal spectra induced by ϕ recovers ϕ_{Var} under the homeomorphism in (b.).

(d.) This statement follows from two easy facts. First, if T is a commutative ring and $W \subseteq T$ is a multiplicative system, then $\mathrm{Spec}(W^{-1}T) \to \mathrm{Spec}(T)$ induced by the localization map $T \to W^{-1}T$ is homeomorphism of $\mathrm{Spec}(W^{-1}T)$ onto the subset $\{P \in \mathrm{Spec}(T) \mid P \cap W = \emptyset\}$. Second, if $I \subseteq T$ is an ideal, then $\mathrm{Spec}(T/I) \to \mathrm{Spec}(T)$ induced by the quotient map $T \to T/I$ is a homeomorphism of $\mathrm{Spec}(T/I)$ onto the closed subset $\mathbb{V}(I)$.

Now, note that $(R_P/PR_P) \otimes_R S = W^{-1}S/I$ where W is the multiplicative set $\phi(R \setminus P) \subseteq S$ and I is the ideal of $W^{-1}S$ generated by $\phi(P)$. Thus, the map

$$S \to (R_P/PR_P) \otimes_R S$$

(which is just localization at W followed by the quotient map corresponding to I) gives a homeomorphism of $\mathrm{Spec}((R_P/PR_P) \otimes_R S)$ onto the subset

$$\{Q \in \mathrm{Spec}(S) \mid \phi(R \setminus P) \cap Q = \emptyset \text{ and } \phi(P) \subseteq Q\} = \{Q \in \mathrm{Spec}(S) \mid \phi^{-1}(Q) = P\}$$

of $\mathrm{Spec}(S)$, i.e. the fiber over P.

(e.) Suppose that $\phi : R \to S$ is an integral extension of domains such that S is finitely generated as an R-algebra.[1] In other words, S is a finitely generated R-module. Let $\mathfrak{m}$ be a maximal ideal of R. Let $K = R/\mathfrak{m}$ and $I = \phi(\mathfrak{m})S$. Then ϕ induces another module finite extension

$$K \to S/I.$$

First note that every prime ideal $\mathfrak{n}$ of S/I is maximal, since the corresponding quotient is a domain which is a finite dimensional K vector space. Thus, every prime ideal of S/I is both maximal and minimal. Since S/I is Noetherian (it is a finite dimensional K vector space), it has only finitely many minimal prime ideals. But, from (d.), we know that $\operatorname{Spec}(S/I)$ corresponds to the fiber of ϕ over $\{\mathfrak{m}\}$. Thus, ϕ induces a finite-to-one map on maximal spectra.

If R and S are finitely generated algebras over an algebraically closed fields, and X and Y are the affine varieties to which they correspond, respectively, then ϕ corresponds to a finite morphism $Y \to X$. The corresponding geometric statement is that a finite morphism is quasi-finite, i.e. $Y \to X$ has finite fibers.

[1] This additional assumption is necessary. When S is not a finitely generated R-algebra, it is possible to give examples where the induced map on maximal spectra is not finite-to-one.