MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 7 SOLUTIONS

Problem 1. We will assume that the characteristic of k is not two. Let Q be a quadric hypersurface in $\mathbb{P}^n$. Then we have seen previously that Q is projectively equivalent to

$$\mathbb{V}(X_0^2 + \cdots + X_r^2)$$

for some $r \leq n$. The quadratic form $X_0^2 + \cdots + X_r^2$ is nondegenerate if and only if $r = n$. In this case, the singular locus[1] of $\mathbb{V}(X_0^2 + \cdots + X_n^2)$ is given by

$$\mathbb{V}(2X_0, \ldots, 2X_n) = \emptyset.$$

Otherwise, if $r < n$, then $[0 : \cdots : 0 : 1]$ is a singular point of $\mathbb{V}(X_0^2 + \cdots + X_r^2)$. Thus, Q is smooth if and only if the associated quadratic form is nondegenerate.

Problem 2. Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{A}^n$ and $y_1, \ldots, y_m$ the coordinates on $\mathbb{A}^m$. Without loss of generality, we may assume $p = 0$ and $F(p) = 0$. Now, F is the restriction of a morphism $\mathbb{A}^n \to \mathbb{A}^m$ given by

$$(x_1, \ldots, x_n) \mapsto (F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n))$$

for some polynomials $F_1, \ldots, F_m \in k[x_1, \ldots, x_n]$. We first describe the corresponding induced map

$$T_0\mathbb{A}^n \to T_0\mathbb{A}^m.$$

Let $\mathfrak{n}$ be the ideal $\langle x_1, \ldots, x_n \rangle$, and $\mathfrak{m}$ the ideal $\langle y_1, \ldots, y_m \rangle$. We can identify $(T_0\mathbb{A}^n)^*$ with $\mathfrak{n}/\mathfrak{n}^2$, and the images $\overline{x_1}, \ldots, \overline{x_n}$ of $x_1, \ldots, x_n$ form a basis. Similarly, we identify $(T_0\mathbb{A}^m)^*$ with $\mathfrak{m}/\mathfrak{m}^2$, and the images $\overline{y_1}, \ldots, \overline{y_m}$ of $y_1, \ldots, y_m$ form a basis. The dual of the map we seek is induced by pull back, so that

$$\overline{y_j} \mapsto \overline{F_j} = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i}(0)\, \overline{x_i}.$$

Thus, using the dual bases to $\overline{x_1}, \ldots, \overline{x_n}$ and $\overline{y_1}, \ldots, \overline{y_m}$, the matrix of our linear transformation is

$$\left(\frac{\partial F_i}{\partial x_j}(0) \right)$$

the Jacobian matrix of the coordinate functions describing F.

Turning to the problem at hand, we have that $T_0 V \subseteq T_0\mathbb{A}^n$ is simply the subspace

$$\{\phi \in (\mathfrak{n}/\mathfrak{n}^2)^* \mid \phi(\overline{f}) = 0 \text{ for all } f \in \mathbb{I}(V)\}$$

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[1] One can also check this by looking on each of the patches in the standard affine cover of $\mathbb{P}^n$. However, we feel it is easier to use the characterization of the projective tangent space given in part (c.) of Problem 4.

and similarly for $T_0 W \subseteq \mathbb{A}^m$. Since the map $\mathbb{A}^n \to \mathbb{A}^m$ takes V to W, the pull-back of any regular function in $\mathbb{I}(W)$ is in $\mathbb{I}(V)$. Thus, the linear map $T_0\mathbb{A}^n \to T_0\mathbb{A}^m$ takes $T_0 V$ into $T_0 W$. This is the induced linear map on tangent spaces $d_0 F : T_0 V \to T_0 W$.

In conclusion, the Jacobian matrix of the coordinate functions describing F determines a linear map $T_0\mathbb{A}^n \to T_0\mathbb{A}^m$. This linear transformation takes the subspace $T_0 V \subseteq T_0\mathbb{A}^n$ into the subspace $T_0 W \subseteq T_0\mathbb{A}^m$, giving the induced linear map on tangent spaces $d_0 F : T_0 V \to T_0 W$.

Problem 3. The varieties $V = \mathbb{V}(xy, yz, xz) \subseteq \mathbb{A}^3$ and $W = \mathbb{V}(xy(x-y)) \subseteq \mathbb{A}^2$ are not isomorphic. If they were, the tangent spaces of corresponding points would have the same dimension. However, direct calculation shows $T_0 V = \mathbb{A}^3$ has dimension 3, while the dimension of the tangent space to W at any point is at most 2.

Problem 4. First, recall that if F is a homogeneous polynomial on $\mathbb{P}^n$ of degree d, we have the Euler formula:

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} x_i = d\,F.$$

Now, suppose $p \in V$ and $\tilde{P} = (\tilde{p}_0, \dots, \tilde{p}_n)$ is any $(n+1)$-tuple representing p. Consider first the definition given in (d.). We have that $q = [q_0 : \cdots : q_n] \in \mathbb{P}(T_{\tilde{P}}\tilde{V})$ if and only if

$$\sum_{i=0}^{n} \frac{\partial g_j}{\partial x_i}(\tilde{P}) \cdot (q_i - \tilde{p}_i) = 0$$

for each $j = 1, \dots, r$. Because of the Euler formula, this is equivalent to the homogeneous linear equations

$$\sum_{i=0}^{n} \frac{\partial g_j}{\partial x_i}(\tilde{P}) \cdot q_i = 0$$

for $j = 1, \dots, r$ and is manifestly independent of the chosen representative for q. Since each $\frac{\partial g_j}{\partial x_i}$ is homogeneous as well, it is independent of the chosen representative $\tilde{P}$ for p. Further, these are precisely the equations described in (c.). Thus, we see that the definitions in (c.) and (d.) are equivalent.

To show the equivalence of definition (b.), we will assume $p = [1 : 0 : \cdots : 0]$ and consider the patch U_0. Let $f_j(x_1, \dots, x_n) = g_j(1, x_1, \dots, x_n)$. The affine variety $V \cap U_0$ is defined by the equations $f_1, \dots, f_r$, and its tangent space at p is given by the equations

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(0, \dots, 0) \cdot x_i = 0$$

for $j = 1, \dots, r$. These are already linear homogeneous polynomials, thus the closure of $T_0(V \cap U_0)$ in projective space is also given by the same equations. We have that $\frac{\partial f_j}{\partial x_i}(0, \dots, 0) = \frac{\partial g_j}{\partial x_i}(1, 0, \dots, 0)$ for $j = 1, \dots, r$ and $i \neq 0$. By the Euler formula once more, we also have $\frac{\partial g_j}{\partial x_0}(1, 0, \dots, 0) = 0$ for

$j = 1, \dots, r$. Thus, the closure of $T_0(V \cap U_0)$ in $\mathbb{P}^n$ is given by the equations

$$\sum_{i=0}^{n} \frac{\partial g_j}{\partial x_i}(1, 0, \dots, 0) \cdot x_i = 0$$

for $j = 1, \dots, r$, which shows the equivalence of (c.) and (d.) with (b.). Further, since the closure of $T_0(V \cap U_0)$ is a linear subvariety of projective space, these are also equivalent with (a.).

Problem 5. On $\mathbb{P}^n \times \mathbb{P}^n$, let $x_0, \dots, x_n$ be the coordinates on the first $\mathbb{P}^n$ and $y_0, \dots, y_n$ the coordinates on the second $\mathbb{P}^n$. Let $g_1, \dots, g_r$ be homogeneous generators of the radical ideal of V. Then

$$V \times \mathbb{P}^n = \mathbb{V}(\langle g_j \,|\, j = 1, \dots, r \rangle)$$
$$TV = \mathbb{V}(\langle \sum_{i=0}^{n} \frac{\partial g_j}{\partial x_i}(x_0, \dots, x_n) \cdot y_i \,|\, j = 1, \dots, r \rangle) \cap (V \times \mathbb{P}^n)$$

is a Zariski closed subset of $V \times \mathbb{P}^n$. Consider the projection $\pi : TV \to V$ onto the first factor. If $p \in V$, then $\pi^{-1}(\{p\}) = \{p\} \times T_pV \simeq T_pV$. If V is smooth of dimension d, then $T_pV \simeq \mathbb{P}^d$ for each $p \in V$. Since each fiber is irreducible of the same dimension d, it follows that TV is smooth of dimension $2d$.

Problem 6. (a.) We have that $\pi^{-1}(\{H\}) = (X \cap H) \times \{H\} \simeq X \cap H$. Since X is not a point, $X \cap H$ is nonempty. Thus, π is surjective.

(b.) Let $\dim X = d$. We have

$$\dim(\pi^{-1}(\{H\}) = \dim(X \cap H) = \begin{cases} d & X \cap H = X, \text{ i.e. } X \subseteq H \\ d-1 & X \nsubseteq H \end{cases}.$$

Generically, we have $X \nsubseteq H$ and $\dim(\pi^{-1}(\{H\}) = d - 1$. All of the fibers have this dimension if X is not contained in any hyperplane.

(c.) Let $d = \dim(X)$, and $\dim(\mathbb{P}(V)) = n$. Consider the other projection $\eta : \mathbb{H} \to X$. If $x \in X$, then

$$\eta^{-1}(\{x\}) = \{x\} \times \{H \in \mathbb{P}(V^*) \,|\, x \in H\}$$

which is isomorphic to a hyperplane in $\mathbb{P}(V^*)$. Since each fiber is irreducible of dimension $n - 1$, we conclude that $\mathbb{H}$ is irreducible of dimension $d + n - 1$.

Problem 7. (a.) Suppose there is an open neighborhood $p \in U \subseteq V$ such that U is isomorphic to a locally closed subset of $\mathbb{A}^n$. In other words, we have an embedding $\iota : U \to \mathbb{A}^n$. This induces an embedding $d\iota_p : T_pU = T_pV \to T_{\iota(p)}\mathbb{A}^n = \mathbb{A}^n$, and the embedding dimension of V at p is at most n.

(b.) Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{A}^n$. Suppose $F \in \mathbb{I}(C)$. Write

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i + G(x_1, \ldots, x_n)$$

where $G \in \langle x_1, \ldots, x_n \rangle^2$. From the definition of C, we have

$$0 = F(t^n, \ldots, t^{2n-1}) = \sum_{i=1}^{n} a_i t^{n+i-1} + G(t^n, \ldots, t^{2n-1})$$

as polynomials in t. All of the terms of $G(t^n, \ldots, t^{2n-1})$ have degree at least $2n$, thus we must have $a_i = 0$ for all $i = 1, \ldots, n$. It follows that the tangent space to C at the origin is $\mathbb{A}^n$.

(c.) It follows from (a.) that the curve constructed in (b.) cannot be embedded in $\mathbb{A}^N$ whenever $n > N$.