MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 8 SOLUTIONS

Problem 1. Let x, y, z be the coordinates on $\mathbb{P}^2$ and t the coordinate of $\mathbb{A}^1$. We will assume the characteristic of k is not three. Consider $C = \mathbb{V}(x^3 + y^3 + z^3 + t(x + y + z)^3) \subseteq \mathbb{P}^2 \times \mathbb{A}^1$ with the projection morphism $\pi : C \to \mathbb{A}^1$. This is a family of varieties parametrized by the affine line. The fiber over $\lambda \in \mathbb{A}^1$ is precisely $\pi^{-1}(\{\lambda\}) = C_\lambda \times \{\lambda\} \simeq C_\lambda$.

The singular locus of C_λ is given by $\mathbb{V}(3x^2+3\lambda(x+y+z)^2, 3y^2+3\lambda(x+y+z)^2, 3z^2+3\lambda(x+y+z)^2)$. Suppose $[a : b : c]$ is a singular point. Then we have $a^2 = b^2 = c^2 = -\lambda(a + b + c)^2$. Without loss of generality, we may assume $a = 1$. Then $b = \pm 1$ and $c = \pm 1$. Checking case by case gives that C_λ is singular at $\left\{ \begin{array}{c} [1:1:1] \\ [1:-1:1] \\ [1:1:-1] \\ [1:-1:-1] \end{array} \right\}$ when λ equals $\left\{ \begin{array}{c} -1/9 \\ -1 \\ -1 \\ -1 \end{array} \right\}$. In other words, the locus of points such over which the fibers of π are smooth is $\mathbb{A}^1 \setminus \{-1, -1/9\}$.

Problem 2. (a.) $\tilde{V}$ has at worst an isolated singularity at the origin if and only if, for all $\tilde{p} \in \tilde{V} \setminus \{0\}$, we have $\dim_{\tilde{p}} \tilde{V} = \dim T_{\tilde{p}} \tilde{V}$. If p is the corresponding point of V, this equality is equivalent to $\dim_p V = \dim T_p V$ since we have seen previously that $T_p V = \mathbb{P}(T_{\tilde{p}} \tilde{V})$ and $\dim_{\tilde{p}} \tilde{V} = \dim_p V + 1$. This means that V is smooth at p if and only if $\tilde{V}$ is smooth at $\tilde{p}$. Thus, $\tilde{V}$ has at worst an isolated singularity at the origin if and only if V is smooth.

(b.) Suppose first that V is not contained in any hyperplane. This is equivalent to saying that the homogeneous ideal of V contains no linear forms, and hence no polynomials with a linear term at all. Since this is also the ideal defining $\tilde{V}$, we see immediately that $T_0 \tilde{V} = \mathbb{A}^{n+1}$.

More generally, let W be the smallest linear subspace of $\mathbb{P}^n$ containing V. The above calculation shows that $T_0 \tilde{V} = \tilde{W} \subseteq \mathbb{A}^{n+1}$. If $V \neq W$, then we have $\dim V < \dim W$ and thus also $\dim \tilde{V} < \dim \tilde{W}$, showing that $\tilde{V}$ is singular at the origin. Thus, $\tilde{V}$ is nonsingular at the origin if and only if V is a linear subvariety of $\mathbb{P}^n$.

(c.) We have seen previously that, if $p \in V$ is represented by $\tilde{p} \in \mathbb{A}^{n+1} \setminus \{0\}$, the projective tangent space $T_p V$ is the projectivization of the linear space given by the kernel of the matrix $\left(\frac{\partial F_i}{\partial x_j}(\tilde{p}) \right)$. Thus, if the rank of this matrix is r, we have that the dimension of $T_p V$ is $n - r$. Since each component of V has dimension $n - c$ by assumption, we know that $T_p V$ always has dimension at least $n - c$. In other words, $r \leq c$ and V is nonsingular at p if and only if $r = c$, i.e. the determinant of some $c \times c$ minor of $\left(\frac{\partial F_i}{\partial x_j}(\tilde{p}) \right)$ does not vanish. It follows immediately that

$$Sing(V) = V \cap \mathbb{V}\left(\left\langle c \times c \text{ minors of } \left(\frac{\partial F_i}{\partial x_j} \right) \right\rangle \right).$$

Problem 3. (a.) Let $\pi : \mathcal{X} \to \mathbb{P}(\mathrm{Sym}^d(V^*))$ be the projection morphism. This is surjective, and the fiber over a point $[F_d] \in \mathbb{P}(\mathrm{Sym}^d(V^*))$, where $F_d \in \mathrm{Sym}^d(V^*)$ is a degree d homogeneous form on V, is $\pi^{-1}([F_d]) = \{[F_d]\} \times \mathbb{V}(F_d) \simeq \mathbb{V}(F_d)$. In other words, the fiber over $[F_d]$ is the hypersurface defined by F_d in V. This allows us to think of the set of hypersurfaces of degree d as a family in the technical sense of the word.

(b.) Consider the other projection $\eta : \mathcal{X} \to \mathbb{P}(V)$. The fiber over $p \in \mathbb{P}(V)$ is isomorphic to the set of hypersurfaces of degree d passing through p. We have already seen that this is a hyperplane in $\mathbb{P}(\mathrm{Sym}^d(V^*))$. Since all of the fibers are irreducible of the same dimension, we see $\mathcal{X}$ is irreducible.

(c.) We have that

$$\Sigma = \mathcal{X} \cap \mathbb{V}\left(\sum_{\substack{x^I = x_0^{i_0}\cdots x_n^{i_n} \\ \text{with } i_j \neq 0}} i_j \cdot a_I \cdot \frac{x^I}{x_j} \quad \text{for} \quad j = 0, \ldots, n \right).$$

In other words, Σ is defined by the equation defining $\mathcal{X}$ and all of its formal partial deriviatives with respect to the variables $x_0, \ldots, x_n$. It is certainly proper: there exist irreducible hypersurfaces of any degree d, and each of those has a smooth point.

(d.) The locus $\pi(\Sigma)$ is precisely the set of points in $\mathbb{P}(\mathrm{Sym}^d(V^*))$ parametrizing singular hypersurfaces of degree d. Since a generic hypersurface of degree d is smooth, we cannot have $\pi(\Sigma)$ is all of $\mathbb{P}(\mathrm{Sym}^d(V^*))$.

Problem 4. (a.) Without loss of generality, we will assume $H = \mathbb{V}(x_0)$ and $p = [0 : \cdots : 0 : 1]$. We have that $T_pV = \mathbb{V}(\sum_{i=0}^n \frac{\partial F}{\partial x_i}(0, \ldots, 0, 1) \cdot x_i)$. Thus, H is tangent to V at p if and only if $\frac{\partial F}{\partial x_i}(0, \ldots, 0, 1) = 0$ for all $i \neq 0$. Let $f(x_1, \ldots, x_n) = F(0, x_1, \ldots, x_n)$. We have $\frac{\partial F}{\partial x_i}(0, \ldots, 0, 1) = \frac{\partial f}{\partial x_i}(0, \ldots, 0, 1)$ for all $i \neq 0$. Hence, H is tangent to V at p if and only if $\frac{\partial f}{\partial x_i}(0, \ldots, 0, 1) = 0$ for all $i \neq 0$, which says precisely that $V \cap H$ is singular at p.

(b.) Let $\mathbb{P}^n = \mathbb{P}(W)$. Consider the map

$$V \to \mathbb{P}(W^*)$$
$$p \mapsto [\frac{\partial F}{\partial x_0}(p) \cdot x_0 + \cdots + \frac{\partial F}{\partial x_n}(p) \cdot x_n]$$

which is well-defined since V is smooth. If we think of $\mathbb{P}(W^*)$ as the set of hyperplanes in $\mathbb{P}(W)$, this map takes p to its corresponding tangent hyperplane. Thus, the image is the set of hyperplanes H which are tangent to V at some point of $V \cap H$. From above, we know this the precisely the set of hyperplanes H such that $H \cap V$ are singular. Thus, what we need to show is that the image is a proper closed subset of $\mathbb{P}(W^*)$. Since V is projective, it is certainly closed; to see that it is proper, note that $\dim(V) = n - 1 < n = \dim(\mathbb{P}(W^*))$.

Problem 5. (a.) $\mathbb{G}_{d+1}(V)$ parametrizes $d+1$ dimensional vector subspaces W of V. To each, we can associate the $n-d$ dimensional vector subspace $(V/W)^*$ of V^* consisting of the linear functionals on V which vanish on W. This gives a natural set-theoretic bijection between $\mathbb{G}_{d+1}(V)$ and $\mathbb{G}_{n-d}(V^*)$. It requires a fair amount of linear algebra to show this is actually a natural isomorphism of varieties. We shall outline it below for completeness.

First, recall without proof that there is a natural isomorphism between $\bigwedge^{n-d}(V^*)$ and $(\bigwedge^{n-d} V)^*$. Further, the morphism

$$\bigwedge^{d+1} V \times \bigwedge^{n-d} V \to \bigwedge^{n+1} V$$
$$(\omega, \eta) \mapsto \omega \wedge \eta$$

is a nondegenerate bilinear pairing. Choosing a basis for V identifies $\bigwedge^{n+1} V$ with k and, using the pairing above, gives an isomorphism between $(\bigwedge^{n-d} V)^* \simeq \bigwedge^{n-d}(V^*)$ and $\bigwedge^{d+1} V$. This isomorphism is not natural! A different choice of basis will alter our isomorphism by a nonzero scalar (the determinant of the change of basis matrix). However, the corresponding linear isomorphism

$$\mathbb{P}(\bigwedge^{n-d}(V^*)) \to \mathbb{P}(\bigwedge^{d+1} V)$$

is natural because it removes this ambiguity. It is now immediate to check that this isomorphism identifies the points in $\mathbb{P}(\bigwedge^{n-d}(V^*))$ represented by primitive vectors with the points in $\mathbb{P}(\bigwedge^{d+1} V)$ represented by primitive vectors, i.e. it maps $\mathbb{G}_{n-d}(V^*)$ isomorphically onto $\mathbb{G}_{d+1}(V)$. Specifically, if $e_1, \ldots, e_{n+1}$ is any basis for V, we have that

$$[e_{d+2}^* \wedge \cdots \wedge e_{n+1}^*] \mapsto [e_1 \wedge \cdots \wedge e_{d+1}].$$

This also shows that our algebraic isomorphism agrees with the set-theoretic bijection described above.

(b.) Choose coordinates identifying $\mathbb{P}(V)$ with $\mathbb{P}^n$. Let $F_1, \ldots, F_r$ be homogeneous generators of the radical ideal defining X. If $\tilde{p} \in V \setminus \{0\}$ represents $p \in X$, then we know

$$T_pV = \mathbb{V}(\sum_{j=0}^{n} \frac{\partial F_1}{\partial x_j}(\tilde{p}) \cdot x_j, \ldots, \sum_{j=0}^{n} \frac{\partial F_r}{\partial x_j}(\tilde{p}) \cdot x_j)$$

Since X is smooth, we know that $\dim T_pV = d$ and hence the linear span of

$$\sum_{j=0}^{n} \frac{\partial F_1}{\partial x_j}(\tilde{p}) \cdot x_j, \ldots, \sum_{j=0}^{n} \frac{\partial F_r}{\partial x_j}(\tilde{p}) \cdot x_j$$

in V^* has dimension precisely $n-d$ (see also Problem 2 (c.)). The function

$$X \to \mathbb{G}_{n-d}(V^*)$$

which associates to p this linear span is independent of the choice of $\tilde{p}$, and is clearly a morphism of varieties. Composing further with the identification of $\mathbb{G}_{n-d}(V^*)$ with $\mathbb{G}_{d+1}(V)$ gives the morphism

ρ and shows it is a morphism of varieties as well. Note that two distinct points p and q in X have the same image under ρ if and only if the d-plane corresponding to their image point is tangent to X at both p and q, i.e. we have a bi-tangent d-plane.

(c.) We will assume the characteristic of k does not divide d (so that X is smooth). If $\mathbb{P}^2 = \mathbb{P}(V)$, then we will use the identification of $\mathbb{G}_2(V)$ with $\mathbb{G}_1(V^*) \simeq \mathbb{P}(V^*) \simeq \mathbb{P}^2$ to describe the map ρ explicitly. We have

$$\begin{aligned} \rho : X &\to \mathbb{P}^2 \\ [a : b : c] &\mapsto [a^{d-1} : b^{d-1} : c^{d-1}]. \end{aligned}$$

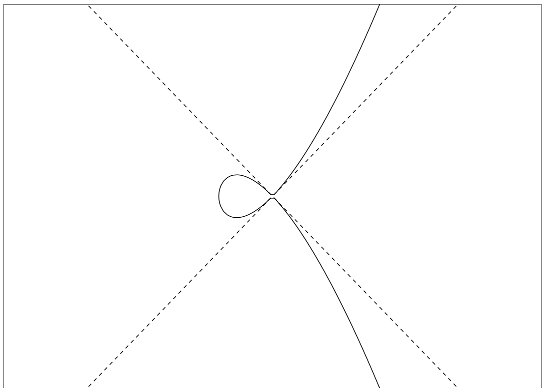
If $d = 1$, the image is a point. Otherwise, this map is a non-constant finite morphism and the image has dimension $\dim(X) = 1$. If $d = 2$, ρ is simply the inclusion of X in $\mathbb{P}^2$. If $d = 3$, the image of the Gauss map consists of points of the form $[a^2 : b^2 : c^2]$ where $a^3 + b^3 + c^3 = 0$. So these are points $[x : y : z]$ where $x^{3/2} + y^{3/2} + z^{3/2} = 0$. Now, using some (high school) algebra, we see that such points satisfy $(x^3 - y^3 - z^3)^2 - 4y^3z^3$, or written more symmetrically, $x^6 + y^6 + z^6 - 2x^3y^3 - 2y^3z^3 - 2x^3z^3$. The image of the Gauss map is defined by this curve because it is irreducible. The polynomial is seen to be irreducible, for instance, by thinking of it as a polynomial in x over $k[y, z]$, it is $x^6 - 2(y^3 + z^3)x^3 + (y^3 - z^3)^2$, which would have to factor as $(x^3 - r)(x^3 - s)$ for some $r, s \in k[y, z]$, but doesn't, eg, by the quadratic formula.

Problem 5. (a.) Denote the tangent cone to the affine variety $V \subseteq \mathbb{A}^n$ at p by C_pV. We know that $C_pV \subseteq T_pV$, and since V is smooth at p both have the same dimension equal to $\dim_p V$. Since T_pV is irreducible, we must have $C_pV = T_pV$ as desired.

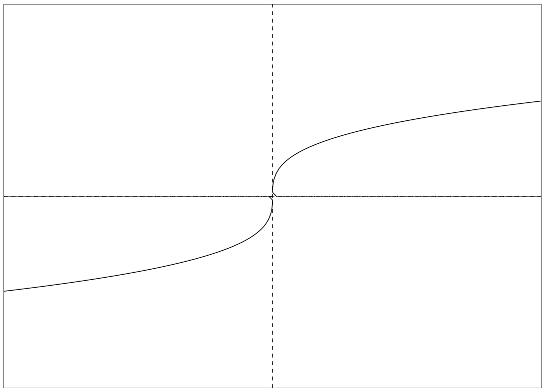
(b.) Say that $V = \mathbb{V}(F(x, y)) \subseteq \mathbb{A}^2$ is a plane curve. We may choose coordinates on $\mathbb{A}^2$ such that p is the origin. If we write $F(x, y) = F_r(x, y) + F_{r+1}(x, y) + \cdots + F_s(x, y)$ where each F_i is a homogeneous polynomial of degree i and $F_r \neq 0$, then $C_p(V) = \mathbb{V}(F_r(x, y))$. Since k is algebraically closed, $F_d(x, y)$ splits as a product of linear factors. Say $F_d(x, y) = \prod_j (a_jx + b_jy)^{l_j}$. Then

$$C_pV = \mathbb{V}(F_d) = \mathbb{V}(\prod_j (a_jx + b_jy)^{l_j}) = \bigcup_j \mathbb{V}(a_jx + b_jy)$$

is a union of lines, as desired. In the graphs that follow, the variety is a solid line, and the tangent cones are dashed lines. If $V = \mathbb{V}(y^2 - x^2 - x^3)$, we have $C_0V = \mathbb{V}(y - x) \cup \mathbb{V}(y + x)$, which looks like:



If $V = \mathbb{V}(xy - y^4)$, we have $C_0V = \mathbb{V}(x) \cup \mathbb{V}(y)$, which looks like:



(d.) In a very small Euclidean neighborhood of a point on a plane curve over the complex numbers, the curve is approximated by its tangent cone at that point. So, in a tiny ϵ-neighborhood of the origin, $\mathbb{V}(x^3y - y^3x - xy^{17} + \pi y^{203})$ looks like $\mathbb{V}(x^3y - y^3x) = \mathbb{V}(x) \cup \mathbb{V}(y) \cup \mathbb{V}(x-y) \cup \mathbb{V}(x+y)$. Also, near the origin, $\mathbb{V}(x^2y + xy^2 - x^4 - y^4)$ looks like $\mathbb{V}(x^2y + xy^2) = \mathbb{V}(x) \cup \mathbb{V}(y) \cup \mathbb{V}(x+y)$.

(e.) Choosing coordinates on $\mathbb{A}^2$ such that p is the origin, we could take

$$V = \mathbb{V}\left(\prod_{i=1}^{k}(y - m_i x) + G(x,y)\right)$$

where $G(x,y)$ is any polynomial with only terms of degrees greater than k. For instance, one could use $G(x,y) = 0$, or $G(x,y) = \pi \cdot x^{1{,}000{,}000{,}000{,}000+k} + \sqrt{-1}y^k$.

(f.) The tangent cone to $V = \mathbb{V}(x^3 - y^2)$ at the origin is $\mathbb{V}(y^2)$. Although this is simply the x-axis as a variety, we would like to think of it as a scheme so that we can consider it as a double line (i.e. the scheme corresponding to the ring $k[x,y]/\langle y^2\rangle$). For instance, this captures that V is not smooth at the origin.