MATH 631: ALGEBRAIC GEOMETRY: HOMEWORK 9 SOLUTIONS

Problem 1. Consider the blowup of the ideal $I = \langle f \rangle \subseteq k[X]$. This is simply the graph of the rational map

$$X \dashrightarrow \mathbb{P}^0$$

$$x \mapsto [f(x)]$$

But $\mathbb{P}^0 = \{*\}$ is simply a point, so this is actually the regular map

$$X \to \{*\}$$

$$x \mapsto *$$

whose graph is $X \times \{*\} \simeq X$.

Problem 2. We have $X = \mathbb{V}(x^2 b - y^3 a) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$, where x, y are the coordinates of $\mathbb{A}^2$ and a, b are homogeneous coordinates for $\mathbb{P}^1$. Consider the affine cover $U = X \setminus \mathbb{V}(a)$ and $V = X \setminus \mathbb{V}(b)$. Then

$$k[U] = k[x, y, b]/(x^2 b - y^3).$$

Thus, the singular locus of U is $\mathbb{V}(2xb, 3y^2, x^2) = \mathbb{V}(x, y) \subseteq U$. Also, we have

$$k[V] = k[x, y, a]/(x^2 - y^3 a).$$

The singular locus of V is $\mathbb{V}(2x, 3y^2 a, y^3) = \mathbb{V}(x, y) \subseteq V$. In fact, this shows that the singular locus of X is $\mathbb{V}(x, y) = \{(0, 0)\} \times \mathbb{P}^1 \subseteq X$. This is really bad! X is singular along an entire divisor! In particular, X is not even normal!

Problem 3. (a.) Without loss of generality, assume $L = \mathbb{V}(y, z) \subseteq \mathbb{A}^3$, where x, y, z are the coordinates on $\mathbb{A}^3$. Then we have $X = \mathbb{V}(yb - za) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$, where a, b are homogeneous coordinates on $\mathbb{P}^1$. Consider the affine cover $U = X \setminus \mathbb{V}(a)$ and $V = X \setminus \mathbb{V}(b)$. Then

$$k[U] = k[x, y, z, b]/(yb - z) \simeq k[x, y, b]$$

so that $U \simeq \mathbb{A}^3$ is smooth. Similarly,

$$k[V] = k[x, y, z, a]/(y - za) \simeq k[x, z, a]$$

and $V \simeq \mathbb{A}^3$ is smooth.

(b.) Since the rational map

$$\mathbb{A}^3 \dashrightarrow \mathbb{P}^1$$

$$(x, y, z) \mapsto [y : z]$$

is defined outside of $L = \mathbb{V}(y,z)$, it follows that π is an isomorphism over $\mathbb{A}^3 \setminus L$. However, if $(\lambda, 0, 0) \in L$, we have

$$\pi^{-1}(\{(\lambda,0,0)\}) = \mathbb{V}(x-\lambda, y, z) = \{(\lambda,0,0)\} \times \mathbb{P}^1 \subseteq X.$$

Thus, the exceptional locus of π is

$$\pi^{-1}(L) = \mathbb{V}(y,z) = L \times \mathbb{P}^1 \subseteq X$$

and it has dimension two.

(c.) We assume $p = (0,0,0)$ and $\ell = \{(t\alpha, t\beta, t\gamma) \,|\, t \in k\}$. Then

$$\begin{aligned}\pi^{-1}(\ell) &= \{((t\alpha, t\beta, t\gamma), [a:b]) \mid t \cdot (\beta b - \gamma a) = 0\} \\ &= \pi^{-1}(\{(0,0,0)\}) \cup (\ell \times \{[\beta : \gamma]\}).\end{aligned}$$

Hence, we have that the proper transform of ℓ is $\ell \times \{[\beta : \gamma]\}$. The only point of the proper transform lying over the origin is $((0,0,0), [\beta : \gamma])$. Geometrically, this point corresponds to the direction through which ℓ intersects L at p.

Problem 4. Since $(0,0) \in C$, we have

$$\pi^{-1}(C) = \pi^{-1}(\{(0,0)\}) \cup \overline{\pi^{-1}(C \setminus \{(0,0)\})}.$$

Note that $E := \pi^{-1}(\{(0,0)\}) = \{(0,0)\} \times \mathbb{P}^1 \simeq \mathbb{P}^1$. Further, recall that the morphism $\pi : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ is an isomorphism over $\mathbb{A}^2 \setminus \{(0,0)\}$. Thus, $\tilde{C} = \overline{\pi^{-1}(C \setminus \{(0,0)\})}$ is an irreducible curve which is birational to C. In short, $\pi^{-1}(C)$ is a reducible curve whose irreducible components are $\tilde{C}$ and E.

We have $\tilde{\mathbb{A}}^2 = \mathbb{V}(xb - ya) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$, where x, y are the coordinates on $\mathbb{A}^2$ and a, b are the homogeneous coordinates on $\mathbb{P}^1$. Consider the affine patch $U = \tilde{\mathbb{A}}^2 \setminus \mathbb{V}(a) \simeq \mathbb{A}^2$. We have that

$$k[U] = k[x,y,b]/(xb - y) \simeq k[x,b]$$

$$E \cap U = \mathbb{V}(x,y) \simeq \mathbb{V}(x, xb) = \mathbb{V}(x)$$

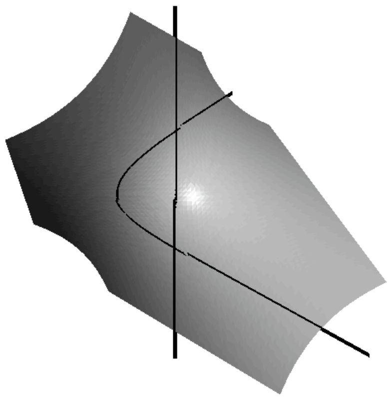
$$\pi^{-1}(C) \cap U = \mathbb{V}(y^2 - x^3 - x^2) \simeq \mathbb{V}(x^2 \cdot (b^2 - x - 1)) = \mathbb{V}(x) \cup \mathbb{V}(b^2 - x - 1).$$

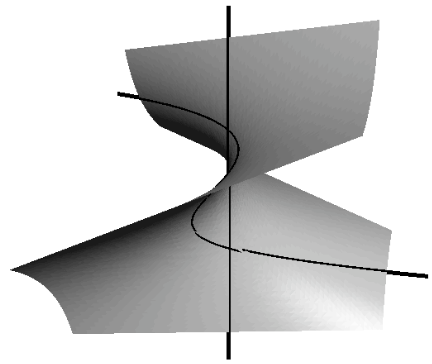
From this, we conclude that $\tilde{C} \cap U$ is isomorphic to the plane curve given by $b^2 - x - 1$, which is smooth. The intersection points of E and $\tilde{C}$ inside U are given by

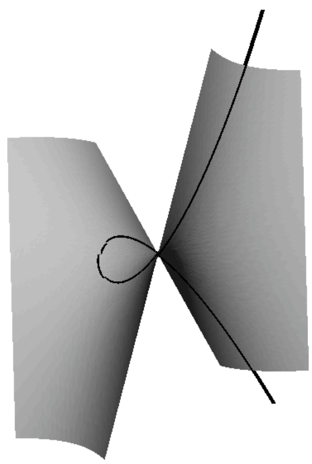
$$\mathbb{V}(x, b^2 - x - 1) = \mathbb{V}(x, b^2 - 1) = \mathbb{V}(x, b-1) \cup \mathbb{V}(x, b+1)$$

which are the points $(0, \pm 1)$ in the coordinates x, b of this affine patch. Similar calculations on the affine patch $V = \tilde{\mathbb{A}}^2 \setminus \mathbb{V}(b)$ show that these points $(((0,0),[1:1])$ and $((0,0),[1:-1]))$ are the only intersection points of E and $\tilde{C}$, and that $\tilde{C}$ is smooth.

Consider the following illustrations on the patch U. In the first two, you can see both components of $\pi^{-1}(C)$. The vertical line is E, whereas the other curve is $\tilde{C}$. The last illustration is a view from the top, and one can visualize the map π as the downward projection.







We know that the points of E correspond to lines through the origin in $\mathbb{A}^2$. The geometric significance of the two intersection points of $\tilde{C}$ and E is that they correspond two the two lines of the tangent cone to C at the origin. The map $\tilde{C} \to C$ can be though of a separating out these two branches of C at the origin: it is one to one off of the origin, but two to one over the origin. If we imagine C as a string folded into a loop on the table, then we are simply lifting the string off of the table so that it no longer crosses itself. However, the map $\pi^{-1}(C) \to C$ additionally creates a whole new string over the crossing point! In particular, the fiber over the origin is not even finite.

Problem 5. (a.) If W_i has codimension c_i, then W_i is the zero set of precisely c_i linear functionals $f_{i1}, \ldots, f_{ic_i}$ on V. But then $\cap_i W_i$ is the zero set of the $\sum c_i$ linear functionals $f_{11}, \ldots, f_{tc_t}$, which has codimension at most $\sum c_i$ (precisely this sum if and only if the f_{ij} are linearly independent in V^*).

Note that, if $t > n = \dim(V)$ and all of the W_i are proper subspaces, equality cannot hold and thus the W_i cannot intersect transversely. If $n = 2$ and $\dim(W_i) = 1$ for all i, then W_i intersect transversely if and only if $i = 2$ and $W_1 \neq W_2$.

(b.) Without loss of generality, we may assume $W \subseteq V \subseteq \mathbb{A}^n$ and p is the origin. Then we have $\mathbb{I}(V) \subseteq \mathbb{I}(W)$, and it follows

$$T_pW = \mathbb{V}(\{df \mid f \in \mathbb{I}(W)\}) \subseteq \mathbb{V}(\{df \mid f \in \mathbb{I}(V)\}) = T_pV$$

(c.) For each i, we have $\dim W_i \leq \dim T_pW_i$ with equality if and only if W_i is smooth at p. Since V is smooth at p, we have

$$\operatorname{codim} T_pW_i = \dim T_pV - \dim T_pW_i = \dim V - \dim T_pW_i \leq \dim V - \dim W_i = \operatorname{codim} W_i.$$

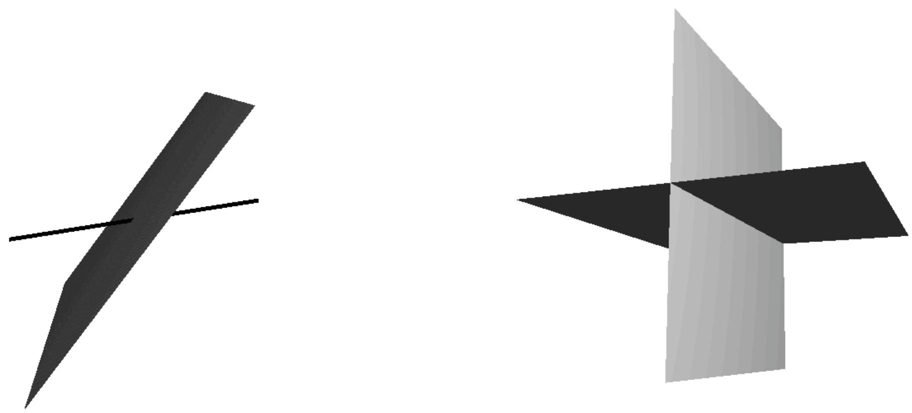
Thus, it follows

$$\operatorname{codim}(\bigcap_{i=1}^{t} T_pW_i) \leq \sum_{i=1}^{t} \operatorname{codim} T_pW_i \leq \sum_{i=1}^{t} \operatorname{codim} W_i.$$

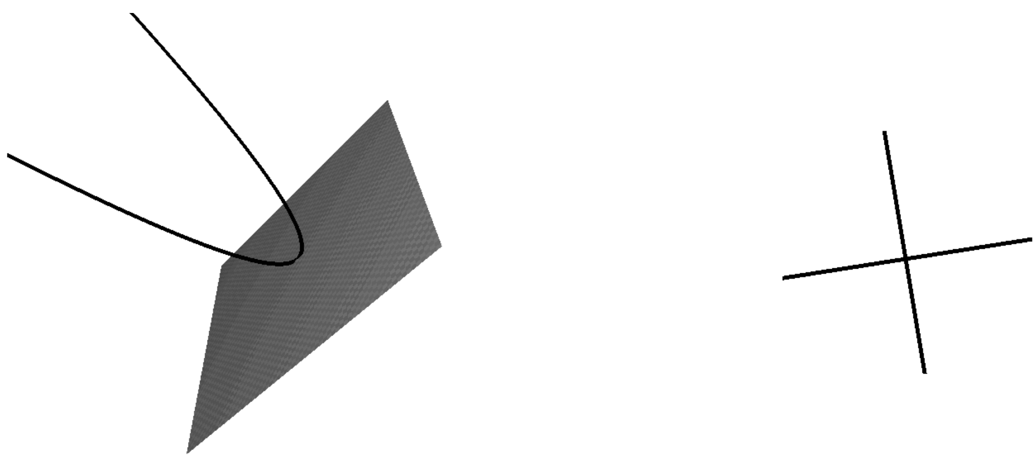
Transversality is equivalent to equality throughout. The first inequality says that the vector subspaces T_pW_i intersect transversely in T_pV, while the second is equivalent to $\dim W_i = \dim T_pW_i$ for all i. Thus, the W_i intersect transversely at p if and only if their they are smooth at p and their tangent spaces intersect transversely in T_pV.

(d.) All varieties drawn are in affine three space. Note that the last one *would* be transverse in two-space.

Transverse



Not Transverse



(e.) Without loss of generality, we may assume V is affine, $u_1, \ldots, u_d$ are regular functions on V, and the ideal $\mathfrak{m}_p$ of regular functions vanishing at p is generated by $u_1, \ldots, u_d$. The Zariski cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$ has as a basis $\bar{u}_1, \ldots, \bar{u}_d$, and we denote the corresponding dual basis of T_pV by $\bar{u}_1^*, \ldots, \bar{u}_d^*$. Let $W_i = \mathbb{V}(\{u_j\}_{j \in J_i})$, where $J_1, \ldots, J_r$ are disjoint subsets of $\{1, \ldots, d\}$. We have that W_i is smooth at p with $\operatorname{codim} W_i = |J_i|$. Also,

$$T_pW_i = \{v \in T_pV \mid v(\bar{u}_j) = 0 \text{ for } j \in J_i\}$$

which has a basis $\{\bar{u}_j^*\}_{j \notin J_i}$. In particular, $\bigcap_{i=1}^r T_pW_i$ has a basis $\{\bar{u}_j^*\}_{j \notin (\bigcup_{i=1}^r J_i)}$, so that

$$\operatorname{codim}(\bigcap_{i=1}^r T_pW_i) = \left|\bigcup_{i=1}^r J_i\right|.$$

Since the J_i are disjoint, we have

$$\left|\bigcup_{i=1}^{r} J_i\right| = \sum_{i=1}^{r} |J_i| = \sum_{i=1}^{r} \operatorname{codim} W_i$$

which implies that the W_i intersect transversely at p.

Problem 6. (a.) Without loss of generality, $p = [1 : 0 : \cdots : 0] \in X$. Let $U = \mathbb{P}^n \setminus \mathbb{V}(x_0) \simeq \mathbb{A}^n$, and let $f(x_1, \ldots, x_n)$ be the local equation of X on U. By definition, X has multiplicity r at p if and only if $f = f_r + f_{r+1} + \cdots + f_s$ where f_j is homogeneous of degree j and $f_r \neq 0$. In particular, X has multiplicity 1 at p if and only if f has a nonzero linear term, i.e. X is smooth at p.

(b.) If ℓ is the line through p and $(a_1, \ldots, a_n) \in U \simeq \mathbb{A}^n$, then we have

$$\begin{aligned} f(ta_1, \ldots, ta_n) &= f_r(ta_1, \ldots, ta_n) + \cdots + f_s(ta_1, \ldots, ta_n) \\ &= t^r \cdot \left(f_r(a_1, \ldots, a_n) + t f_{r+1}(a_1, \ldots, a_n) + \cdots + t^{s-r} f_s(a_1, \ldots, a_n)\right) \end{aligned}$$

which shows that the multiplicity of X along ℓ is at least r.

(c.) We can see this in two ways. Let $p \in X$ be a point of multiplicity r. Choose a line ℓ in $\mathbb{P}^n$ through p which does not lie entirely on X. Recall from Problem 4 (a.) on Problem Set 6 that ℓ intersects X in precisely d points (counted according to multiplicity). From (b.), it follows immediately that $r \leq d$.

Alternatively, again we may assume $p = [1 : 0 : \cdots : 0] \in X$ and $U = \mathbb{P}^n \setminus \mathbb{V}(x_0) \simeq \mathbb{A}^n$ as above. Let $F(x_0, \ldots, x_n)$ be the homogeneous polynomial defining X. Then $f(x_1, \ldots, x_n) = F(1, x_1, \ldots, x_n)$ is the local equation for X on U. Since F is homogeneous of degree d, we see immediately that f has degree at most d. In particular, the smallest order terms of f have degree at most d, which says that X has multiplicity at most d at p.

(d.) Suppose that ℓ is any line through p. If ℓ is not contained in X, then (from Problem 4 (a.) on Problem Set 6) ℓ intersects X in precisely d points (when counted with multiplicity). Thus, using part (b.), it follows that p is the only intersection point of X and ℓ, and we see that X is a cone with vertex at p.

(e.) Again we may assume $p = [1 : 0 : \cdots : 0] \in X$ and $U = \mathbb{P}^n \setminus \mathbb{V}(x_0) \simeq \mathbb{A}^n$. Let $F(x_0, \ldots, x_n)$ be the homogeneous polynomial of degree d defining X. By assumption, we have

$$F(x_0, \ldots, x_n) = x_0 f_{d-1}(x_1, \ldots, x_n) - f_d(x_1, \ldots, x_n)$$

where f_{d-1} and f_d are nonzero homogeneous polynomials of degrees $d-1$ and d, respectively. Consider the projection morphism away from the point p:

$$\theta : X \dashrightarrow \mathbb{P}^{n-1}$$

$$[x_0 : \cdots : x_n] \mapsto [x_1 : \cdots : x_n].$$

Since X has multiplicity $d-1$ at p, every line through p which is not contained in X must intersect X in precisely one point away from p. This implies that θ is injective off of the lines through p contained in X. (Since p is not a cone point there is some line through p not on X). In fact, we can see that θ is birational by explicitly constructing its rational inverse

$$\begin{aligned} \psi : \mathbb{P}^{n-1} &\dashrightarrow X \\ [x_1 : \cdots : x_n] &\mapsto [\frac{f_d(x_1, \ldots, x_n)}{f_{d-1}(x_1, \ldots, x_n)} : x_1 : \cdots : x_n] \\ &= [f_d(x_1, \ldots, x_n) : x_1 f_{d-1}(x_1, \ldots, x_n) : \cdots : x_n f_{d-1}(x_1, \ldots, x_n)]. \end{aligned}$$