Let \( p \) be a (closed) point on an irreducible variety \( V \) of dimension \( d \) over an algebraically closed field \( k \). We will define some invariants of this (possibly non-singular) “singularity.”

Let \((R, m, k)\) be the local ring of \( V \) at \( p \). For each natural number \( n \), the \( R \)-module \( R/m^n \) is a finite dimensional vector space over \( k \); its dimension, as a function of \( n \), is called the Hilbert-Samuel function. For large \( n \), the Hilbert-Samuel function is eventually a polynomial of degree \( d \), called the Hilbert-Samuel polynomial. Its leading coefficient (normalized, by convention, by multiplying by \( d! \)) is called the Hilbert Samuel multiplicity of \( p \).

That is, the Hilbert Samuel multiplicity of \( p \) is
\[
d! \lim_{n \to \infty} \frac{\dim_k (R/m^n)}{n^d}.
\]
It is not too hard to prove that the Hilbert-Samuel multiplicity of a smooth point \( p \) is 1. In general, the Hilbert-Samuel multiplicity is a positive integer equal to 1 if and only if \( p \) is smooth. Larger multiplicities indicate “more singular” points.

In this worksheet, you will compare the Hilbert-Samuel multiplicity to a characteristic \( p \) multiplicity called Hilbert-Kunz multiplicity.

Assume now that the ground field \( k \) has prime characteristic \( p \). Otherwise, \( R, V, \) and \( p \) are as above.

Warmup: Compute, directly, the Hilbert Samuel multiplicity of the origin in \( k^2 \).

Problem 1: Show that The generic rank of the \( R \)-module \( F_c^e R \) is \( p^d e \). By definition, this means that \( K \otimes_R F_c^e R \) has dimension \( p^d e \) over \( K \), where \( K \) is the function field of \( V \) (=fraction field of \( R \)). You may use the following steps:

1. Prove that the generic rank of \( F_c^e R \) over \( R \) equals \([K : K^{p^n}]\).
2. Let \( x_1, \ldots, x_d \) be a separating transcendence basis for \( K/k \). Show that \( L^{1/p} \otimes_L K \cong L^{1/p} K = K^{1/p} \). [Recall, by definition of separating transcendence basis, the field extension \( K/k \) factors as
\[
k \subset k(x_1, \ldots, x_d) = L \subset K
\]
where the first extension is purely transcendental and the second is separable.] [Hint: For the first \( \cong \), use the definition of separable to show \( L^{1/p} \otimes_L K \) is a field; For the second \( = \), put all the fields into an inclusion diagram and consider the degree of each extension.]

Problem 2: The minimal number of generators for the \( R \)-module \( F_c^e R \) is \( \dim_k (R/m^{|c|}) \).

1. Prove this statement using Nakayama’s Lemma. [We did this in class, think it through yourself again.]
(2) Show the minimal number of generators for the $R$-module $F^e R$ is bounded below by $p^{de}$. [Hint: Use Fact 1.]

(3) Prove that the point $p$ is smooth if and only if $\dim_k(R/m[pe]) = p^{de}$ for all $e$. [Hint: Use Kunz’s theorem. It might be helpful to remember that for finitely generated modules over a local Noetherian ring, flat is the same as free.]

**Problem 3:** Monsky proved that the function $\dim_k(R/m[pe])$ grows like a polynomial of degree $d$ in $p^e$. So we can define the Hilbert-Kunz multiplicity as

$$\lim_{e \to \infty} \frac{1}{p^{de}} \dim_k(R/m^{[pe]}).$$

(1) The Hilbert-Kunz multiplicity is a measure of the asymptotic growth of the actual minimal number of generators for $F^e R$ compared to the idealized minimal number of generators we’d have in the smooth case. Explain.

(2) Let $\mu$ be the embedding dimension of $R$. Show that $m^{\mu p^e} \subset m^{[pe]}$ for all $e$. [Hint: Consider monomial generators for $m^{\mu p^e}$.]

(3) Show that for all natural numbers $e$, we have

$$\dim_k(R/m^{\mu p^e}) \geq \dim_k(R/m^{[pe]}) \geq \dim_k(R/m^{pe})$$

(4) Writing $HS(p)$ and $HK(p)$ for the Hilbert-Samuel and Hilbert-Kunz multiplicities, respectively, show that

$$\frac{\mu^d}{d!} HS(p) \geq HK(p) \geq \frac{1}{d!} HS(p) \geq \frac{1}{d!}.$$ 

Hilbert-Kunz multiplicities are notoriously hard to calculate. An open question is to find nice hypothesis under which they are rational. An open question is whether or not they are always algebraic numbers (until fairly recently, it had been conjectured that they are rational, but a counter-example of Monsky shows this is false).

There is much more to say about Hilbert-Kunz multiplicities; let me know if you might be interested in reading further and/or giving a talk on them.