Worksheet on $I$-adic Topology and Completion

Let $R$ be a ring and $I \subset R$ an ideal. Let $R^\mathbb{N}$ denote the ring of all sequences $(r_n)_{n \in \mathbb{N}}$ in $R$.

**Definition.** We say a sequence in $R^\mathbb{N}$ is **Cauchy for the $I$-adic topology** if for all $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that whenever $n, m \geq d$, we have $r_n - r_m \in I^t$.

1. Show that the subset $C^I(R)$ of Cauchy sequences in $R^\mathbb{N}$ is a subring.
2. Show that the map $R \to C^I(R)$ sending $r$ to the constant sequence $(r)$ is a ring homomorphism, making $C^I(R)$ into an $R$-algebra.
3. Let $C^0_I(R)$ be the set of sequences $(r_i)_{i \in \mathbb{N}}$ that converge to zero, meaning that for all $n$ there exists $m$ such that for all $i \geq m$, $r_i \in I^n$. Prove that $C^0_I(R)$ is an ideal of $C^I(R)$.
4. Observe that a subsequence of a Cauchy sequence is Cauchy, and differs from the original by a sequence converging to zero.

**Definition.** The **completion of $R$ in the $I$-adic topology** is the ring $\hat{R}^I = C^I(R)/C^0_I(R)$.

5. The elements of $\hat{R}^I$ are equivalence classes of Cauchy sequences that differ by Cauchy sequences converging to zero. We call such an equivalence class a **limit of a Cauchy sequence** in $R$. Compare this to the construction of the real numbers as the collection of limits of Cauchy sequences of rational numbers.
6. Discuss a natural map $R \to \hat{R}^I$. What is its kernel?
7. Suppose $(R, m)$ is Noetherian and local. Prove that $R \to \hat{R}^m$ is injective.
8. Let $R = \mathbb{Z}$ and $I = \langle p \rangle$. Show that

$$(\sum_{i=0}^n p^i)_{n \in \mathbb{N}} = (1, 1 + p, 1 + p + p^2, 1 + p + p^2 + p^3, \ldots)$$

is a Cauchy sequence for the $p$-adic topology on $\mathbb{Z}$. Show also that it represents the inverse of $1 - p$ in the completion $\hat{\mathbb{Z}}^p$ (also denoted $\mathbb{Z}_p$).

9. Let $R = K[x, y]$ and $I = \langle x, y \rangle$.
   a. Show that

$$(\sum_{i=0}^n r_i x^i y^{i+2})_{n \in \mathbb{N}} = (r_0 y^2, r_0 y^2 + r_1 xy^3, r_0 y^2 + r_1 xy^3 + r_2 x^2 y^4, \ldots)$$

is a Cauchy sequence in the $I$-adic topology.
   b. Write a Cauchy sequence different from the one in (a) but which has the same limit (that is, represents the same element of $\hat{R}^I$, or equivalently, differs by a sequence converging to zero).
   c. Describe a natural ring map\(^1 K[[x, y]] \to \hat{R}^I\) taking a power series $\sum_{i=0}^\infty r_{ij} x^i y^j$ to an equivalence class of Cauchy sequences. Can you show it’s injective?

\(^1\) think: successive truncations
(10) Explain how to think of an element of \( \hat{R}^I \) (non-uniquely) as a “formal power series” \( \sum_{n=0}^{\infty} x_n \) where \( x_n \in I^n \). This might help you prove surjectivity in (9c).

**Theorem.** Let \( S = K[x_1, \ldots, x_n] \) and \( I = \langle x_1, \ldots, x_n \rangle \). Then \( \hat{S}^I = K[[x_1, \ldots, x_n]] \). Furthermore, let \( g_1, \ldots, g_t \in K[x_1, \ldots, x_n] \subseteq K[[x_1, \ldots, x_n]] \). Then for \( R = K[x_1, \ldots, x_n] / \langle g_1, \ldots, g_t \rangle \) we have \( \hat{R}^I = K[[x_1, \ldots, x_n]] / \langle g_1, \ldots, g_t \rangle \).

(11) Assume the Theorem. Let \( R = \mathbb{R}[x, y] / \langle y^2 - x^2 - x^3 \rangle \). Let \( m = \langle x, y \rangle \).

(a) Prove that \( R \) is a domain.
(b) Show that \( \hat{R}^m \) is not a domain. [Hint: find a power series representing \( \sqrt{1+x} \).]
(c) Draw a picture of the curve in \( \mathbb{R}^2 \) with coordinate ring \( R \).
(d) There is a sense in which Spec \( \hat{R}^m \) can be considered a very small neighborhood around \( m \in \text{Spec} \ R \). How is this reflected in your picture?

(12) Let \( (r_n)_{n \in \mathbb{N}} \in \mathfrak{C}_I(R) \) be a Cauchy sequence. Fix \( t \in \mathbb{N} \).

(a) Explain why the sequence of residues \( r_n \mod I^t \) is eventually constant as \( n \to \infty \).
(b) Use (a) to show there is a surjective ring homomorphism \( \mathfrak{C}_I \to R / I^t \).
(c) Use (b) to show there is a surjective ring homomorphism \( \hat{R}^I \to R / I^t \).
(d) Show that for any \( t' \leq t \), the map for \( t' \) is the composition \( \hat{R}^I \to R / I^t \to R / I^{t'} \).

**Definition.** A **directed set** is a partially ordered set \((\Lambda, \leq)\) with the property that for any \( \lambda_1, \lambda_2 \in \Lambda \), there exists \( \mu \in \Lambda \) with \( \lambda_i \leq \mu \). View \((\Lambda, \leq)\) as a category whose objects are \( \lambda \in \Lambda \), with exactly one morphism from \( \lambda \) to \( \mu \) if \( \lambda \leq \mu \) (and none otherwise).

**Definition.** A **direct limit system** in a category \( C \) is collection of objects \( X_\lambda \) in \( C \) indexed by \( \Lambda \), together with morphisms \( f_{\lambda \mu} : X_\lambda \to X_\mu \) whenever \( \lambda \leq \mu \). These morphisms satisfy \( f_{\lambda \lambda} \) is the identity on \( X_\lambda \) and, whenever \( \lambda \leq \mu \leq \nu \), we have \( f_{\mu \nu} \circ f_{\lambda \mu} = f_{\lambda \nu} \). Briefly put, a direct limit system in \( C \) is a covariant functor from \((\Lambda, \leq)\) to \( C \). An **inverse limit system** in \( C \) is direct limit system in \( C^{\text{op}} \), or equivalently, a contravariant function from \((\Lambda, \leq)\) to \( C \).

(13) Write out the definition of inverse limit system (in the category of rings, say) explicitly.

(14) Fix a topological space \( X \).

(a) Show that the collection \( \Lambda \) of all open sets of \( X \) is a directed set, with the partial order being inclusion.
(b) Let \( \mathcal{O}_X(U) \) be ring of continuous real-valued functions on \( U \in \Lambda \). Show that restriction induces an inverse limit system indexed by \( \Lambda \) in the category of commutative rings.

(15) Fix a ring \( R \) and an ideal \( I \).

(a) Show that \( \mathbb{N} \) is a directed set.
(b) Show that the natural quotient maps \( R / I^n \to R / I^m \) (whenever \( n \geq m \)) form an inverse limit system.
Definition. Fix a direct limit system in a category $\mathcal{C}$. An object $X$ of $\mathcal{C}$ is its **direct limit** if there are morphisms $g_\lambda : X_\lambda \to X$ commuting with all the maps in the limit system such that the following universal property is satisfied: for any object $Y$ with the property that $X_\lambda \to Y$ compatibly the maps in the limit system, there is a unique morphism $X \to Y$ making all diagrams commute. An **inverse limit** of an inverse limit in $\mathcal{C}$ is the direct limit in $\mathcal{C}^{op}$.

(16) Write out the definition of inverse limit (in the category of rings, say) explicitly.