Let $G$ be the group of rotational symmetries of a regular icosahedron.

1. Be sure you can explain how to compute the order of $G$, as well as the reason why $G$ can also be viewed as the rotational symmetry group of a regular dodecahedron. For reference, it might be helpful to compute the number of vertices, edges and faces of the solids you are dealing with.

2. Compute the number of Sylow 3 and 5-groups of $|G|$ and make sure you can identify these physically in terms of rotations of an actual solid.

3. Compute the following data for $G$:

   1. The number of elements of order 5; describe them.
   2. The number of elements of order 3; describe them.
   3. The number of elements of order 2; describe them. [Hint: consider the action of $G$ on the edges; what is the stabilizer of an edge?]
   4. The number of elements of order 1.

   Explain why this lists all elements of $G$, and why the breakdown into conjugacy classes must be a refinement of these. Explain why this list can not be the breakdown into conjugacy classes.

4. Find the conjugacy classes of $G$ explicitly, and the number of elements in each.\(^1\)

\(^1\)Hints: recall that changing coordinates in $\mathbb{R}^3$ is conjugation by an invertible matrix. To see that all order 2 elements are conjugate, note that $G$ acts transitively on the edges. Note also that $G$ acts transitively on Sylow $p$-groups to restrict the ways (1) and (2) above could refine into classes. Also, because $G$ acts on pairs of opposite faces, we can see that $G$ acts a lot like a dihedral group on a face-pair.
5. Show that $G$ is simple. [Hint: a normal subgroup is a union of conjugacy classes; why?]

6. Show that $G$ has no non-trivial one dimensional representation (use (5)).

7. The icosahedral group comes equipped with a 3-dimensional representation. What is it? It is irreducible (Use 6)?

8. Figure out the number and dimensions of the irreducible complex representations of $G$, by using the fact that the sum of the dimensions of the irreducible reps is $|G|$.

9. By inscribing five (overlapping) cubes in a dodecahedron, show that $G$ is isomorphic to $A_5$. [Hint: There are 5 ways to choose six pairs of opposite edges in dodecahedron.] Use this to confirm your answer to 8.

10. Using the permutation representation induced by (9), find a five dimensional permutation representation, and use it to construct an irreducible four dimensional representation.

11. Note that $G$ acts on the six axes of rotations of the dodecahedron through opposite faces. Use this to construct a six dimensional representation and a five dimensional irreducible representation.

12. Considering $G \cong A_5$ as a subgroup of $S_5$, note that conjugation by an element $\sigma \in S_5$ induces a group automorphism of $A_5$ (and hence $G$). Explain how this might be used in general to construct additional representations of a group from known ones. Use this trick to construct a three dimensional irreducible representation of $G$ not already on your list.

13. Complete the character table of $G$.

<table>
<thead>
<tr>
<th>Classes of $G$</th>
<th>$\chi_{triv}$</th>
<th>$\chi_{taut}$</th>
<th>$\chi_3$</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>#elements in each class</td>
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<td>1</td>
<td></td>
<td></td>
<td></td>
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</tbody>
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