Math 217 §5.5 Professor Karen Smith

INQUIRY: WHAT IS AN INNER PRODUCT SPACES?

Your group is given one vector space, together with a rule for combining two vectors into a scalar. Investigate whether your given rule makes the space into an inner product space.

1. Is it an inner product space? Prove it, or adapt it to make it an inner product space. [One of the three is not.]
2. Pick any non-zero vector in your inner product space. Find its length.
3. Find a pair of orthonormal vectors $\vec{u}_1, \vec{u}_2$ in your space. Let W be the space that they span. What is $\dim W$?
4. Compute the distance between $\vec{u}_1$ and $\vec{u}_2$ in your inner product. Also compute $||\vec{u}_1 - \vec{u}_2||^2$.
5. Find an element $\vec{z}$ in $W^\perp$. What is the closest vector in W to $\vec{z}$? What is the projection of $\vec{z}$ onto W?
6. Find an element $\vec{q}$ not in $W \cup W^\perp$. Compute the projection of $\vec{q}$ to W. What is the closest vector in W to $\vec{q}$?

A. Let C^0 be the space of continuous functions. Let $\langle f, g\rangle$ be defined by $\int_{-1}^{1} fg\,dx$.

Solution note: 1). Yes, this is an inner product. See the book for a proof (we did it in class also).
2). $||1|| = \sqrt{\int_{-1}^{1} 1\,dx} = \sqrt{2}$.
3). Let $f = 1$ and $g = x$. Then $\int_{-1}^{1} fg\,dx = 0$ since x is odd. So this is almost an orthonormal basis: we need only scale each by its length. So let $\vec{u}_1 = 1/\sqrt{2}$ and $\vec{u}_2 = \sqrt{\int_{-1}^{1} x^2\,dx} = \frac{\sqrt{2}}{\sqrt{3}}$. The dimension of W is 2.
4). The distance between is $\sqrt{2}$.
5). For any vector in $W^\perp$, the projection to W is zero. So the closest vector in W is zero as well.
6). let $q = x + 1$. Then the projection onto W is $\frac{1}{2}\langle 1, x+1\rangle 1 + \frac{2}{3}\langle x, x+1\rangle x$. You can compute this by computing the integrals.

B. Let $\mathbb{R}^{2\times 2}$ be the space of 2×2 matrices. Define $\langle A, B\rangle$ by trace $(A^T B)$.

Solution note: 1.) This is an inner product. The linearity holds: since $(A_1+A_2)^T B = A_1^T B + A_2^T B$ and trace is linear, the additivity holds; also since $(kA)^T B = k(A^T B)$ and trace is linear, the scalar multiplication is respected, too. The symmetric property follows from basic properties of trace: the trace is the same for a matrix and its transpose, so trace $(A^T B) =$ trace $(A^T B)^T$. The latter is trace $B^T A = \langle B, A \rangle$.
2). The length of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is $\sqrt{(trace(A^T A))} = \sqrt{(trace(I_2))} = \sqrt{2}$.
3). Let vectors E_{11} and E_{22} are orthnormal, as you can easily check by computing $\langle E_{11}, E_{11} \rangle = 1$, $\langle E_{11}, E_{22} \rangle = 0$ and $\langle E_{22}, E_{22} \rangle = 1$. The space they span has dimension 2.
4). $||u_1 - u_2|| = \sqrt{trace(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})^2} = \sqrt{2}$.
5). The element E_{12} is in $W^\perp$. It's projection to W is zero. Thus zero is the closest vector in W to E_{12}.
6). Let $\vec{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The projection to W is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

C. Let V be $\mathbb{R}^3$. Define $\langle \vec{x}, \vec{y} \rangle$ as the matrix product $\vec{x}^T A \vec{y}$ where A is a diagonal matrix.

Solution note: For this we need A to have strictly positive entries on the diagonal. Otherwise, it is not an inner product because positive definitivity fails.

Bonus: Prove that the distance between a pair of orthonormal vectors in any inner product space is $\sqrt{2}$.