Worksheet on Affine Noetherian Schemes

Let $R$ be a commutative ring with 1.

**Definition:** A topological space is **irreducible** if it cannot be written as the union of two proper closed subspaces. An **irreducible component** of a topological space is a closed irreducible subset which is maximal with respect to inclusion.

**Theorem 1.** The irreducible components of $\text{Spec } R$ are the closed sets $\mathbb{V}(P)$ where $P$ is a minimal prime of $R$, and $\text{Spec } R$ is the union of these irreducible components. In particular, if $R$ is a domain, $\text{Spec } R$ is irreducible.

**Theorem 2.** Every ideal in a Noetherian ring has finitely many minimal primes. In particular, a Noetherian ring has finitely many minimal primes.

**Corollary 2.** If $R$ is Noetherian, then $\text{Spec } R$ has finitely many irreducible components.

(1) Review the following facts about the Zariski topology on $\text{Spec } R$:

(a) The closed sets are $\mathbb{V}([f_\lambda]_{\lambda \in \Lambda}) = \mathbb{V}(\sqrt{\langle \{f_\lambda\}_{\lambda \in \Lambda} \rangle})$.

(b) For any subsets $I \subset J$ of $R$, $\mathbb{V}(J) \subset \mathbb{V}(I)$.

(c) $\bigcap_{\lambda \in \Lambda} \mathbb{V}(I_\lambda) = \mathbb{V}(\bigcup_{\lambda \in \Lambda} I_\lambda)$.

(d) $\bigcup_{i=1}^t \mathbb{V}(I_i) = \mathbb{V}(I_1 \cap \cdots \cap I_t) = \mathbb{V}(I_1 \cdot \cdots \cdot I_t)$.

(e) If $N \subset R$ denotes the nilradical of $R$, then the closed set $\mathbb{V}(N)$ is equal to $\text{Spec } R$.

(f) For any ideal $I$, contraction for the quotient map $R \to R/I$ induces a homeomorphism $\text{Spec } R/I \cong \mathbb{V}(I)$, where $\mathbb{V}(I) \subset \text{Spec } R$ has the subspace topology.

(2) Using the theorems above, which of the following are irreducible topological spaces? Count the components of those that are not.

(a) $\text{Spec } \mathbb{Z}$

(b) $\text{Spec } \mathbb{Z}/\langle 24 \rangle$

(c) $\text{Spec } \mathbb{Z}/\langle 25 \rangle$

(d) $\text{Spec } K[x, y, z, w]/\langle xy^5 + z^3y^2 + wx \rangle$. [Hint: Use Eisenstein with the prime $\langle z, w \rangle$.]

(e) $\text{Spec } K[x, y, z, w]/\langle xy, xz, xw \rangle$.

(f) $\text{Spec}(L_1 \times L_2)$ where $L_1, L_2$ are fields.

(g) $\text{Spec } R$ where $R$ is the product, over all positive prime integers $p$, of the rings $\mathbb{Z}/p\mathbb{Z}$.

(3) **Proof of Theorem 1.** Let $R$ be arbitrary.

(a) Prove that if $\text{Spec } R = \mathbb{V}(x)$, then $x$ is nilpotent.

(b) Suppose that $R$ is reduced. Show that if $\text{Spec } R$ is irreducible, then $R$ is a domain. [Hint: Recall that $\mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$.

(c) Assume $R$ is a domain. Prove that $\text{Spec } R$ is irreducible. [Hint: What closed sets contain $\langle 0 \rangle$?]

(d) Show that every irreducible closed set of $\text{Spec } R$ has the form $\mathbb{V}(P)$ where $P \in \text{Spec } R$.

(e) Prove that the irreducible components of $\text{Spec } R$ are of the form $\mathbb{V}(P)$ where $P$ is a minimal prime of $R$.

(f) Prove that $\text{Spec } R = \bigcup_{P \in \text{minSpec } R} \mathbb{V}(P)$.
(4) **Proof of Theorem 2.** We’ll use Noetherian Induction.

(a) Explain why Theorem 2 is equivalent to the (a priori weaker) statement that every Noetherian ring has finitely many minimal primes.

(b) To prove Theorem 2, fix a Noetherian ring \( R \). Consider the set of all ideals in \( R \) that have infinitely many minimal primes. Show that this set has a maximal element (with respect to inclusion) if it is non-empty.

(c) To prove Theorem 2, show that it suffices to prove it for \( R \) with the property that every proper quotient has finitely many minimal primes. [Hint: Use (b).]

(d) Explain why, if \( R \) in (c) is a domain, the proof of Theorem 2 is complete.

(e) With \( R \) as in (c), suppose \( x, y \in R \) are non-zero elements such \( xy = 0 \). Show that every minimal prime of \( R \) contains either \( x \) or \( y \) (or both).

(f) Again, with \( R \) as in (c), show that \( \langle x \rangle \) and \( \langle y \rangle \) have only finitely many minimal primes. [Hint: Use the Noetherian induction hypothesis (c).]

(g) Prove Theorem 2 and its corollary.

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(5) **Lemma.** Fix a vector space over an infinite field. Let \( \{W_1, \ldots, W_n\} \) be a finite collection of vector subspaces. Prove that if \( V \) is a subspace contained in \( \bigcup_{i=1}^n W_i \), then \( V \subset W_i \) for some \( i \).

[Hint: Say \( x \in V \), and wlog, \( x \in W_1 \). Pick \( y \in V \setminus W_1 \). Consider the elements \( x + ay \) where \( a \in K \setminus \{0\} \). Induce.]

(6) In the ring \( \mathbb{C}[x, y, z] \), let \( U = \mathbb{C}[x, y, z] \setminus (\langle x \rangle \cup \langle y, z \rangle) \). Let \( R = U^{-1}\mathbb{C}[x, y, z] \). Describe Spec \( R \): what are the maximal and minimal primes? How many components? What is the dimension? What are the heights of the different maximal ideals? How does this look as a poset? as a subset of Spec \( \mathbb{C}[x, y, z] \)? What is its closure in Spec \( \mathbb{C}[x, y, z] \)?

[Hint: Use the Lemma in (5)! Don’t forget that ideals in a \( K \)-algebra are also \( K \)-subspaces.]

(7) **Prime Avoidance Lemma.** Let \( R \) be any ring, and let \( I_1, \ldots, I_t \) be ideals of \( R \). Suppose that an ideal \( J \subset I_1 \cup I_2 \cup \cdots \cup I_t \).

(a) If \( R \) is an algebra over an infinite field, prove that \( J \subset I_k \) for some index \( k \). [Hint: Use (5).]

(b) * More generally, assume most two of the ideals \( I_k \) are not prime. Prove that \( J \subset I_k \) for some index \( k \).

(8) * Consider a doubly indexed set of variables \( \{x_{ij} | i \leq j, \ i, j \in \mathbb{N}\} \). Let \( S \) be the polynomial ring they generate over \( \mathbb{C} \), so \( S = \mathbb{C}[x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, x_{33}, \ldots] \). For each fixed \( j \), let \( P_j \) be the prime ideal generated by \( \{x_{1j}, x_{2j}, \ldots, x_{jj}\} \). Let \( U = S \setminus \bigcup_{j=1}^\infty P_j \).

(a) Show that \( U = S \setminus \bigcup_{n=1}^\infty P_n \) is a multiplicative set. Let \( R = U^{-1}S \).

(b) ** Show that if an ideal \( I \subset S \) is contained in \( \bigcup_{n=1}^\infty P_n \), then \( I \subset P_n \) for some \( n \). [Hint: for \( f \in I \), consider the (non empty, finite) set \( Q(f) := \{i \in \mathbb{N} | f \in P_i R\} \). Show we’re done unless \( \forall f \in I, \exists g \in I \) such that \( Q(f) \cap Q(g) = \emptyset \). Now look at \( f + x_m^n g \) (which is in \( I \)) for well-chosen \( m \in \mathbb{Q}(g) \) and \( d \gg 0 \].

(c) Prove that the maximal ideals of \( R = U^{-1}S \) are precisely the \( P_j R \).

(d) Show that \( R \) has chains of primes of arbitrarily long length.

(e) Prove that the localization of \( R \) at any maximal ideal is Noetherian.

(f) Prove that any non-zero \( f \in R \) is contained in at most finitely maximal ideals of \( R \).

(g) * Prove that a ring is Noetherian if its localization at any maximal ideal is Noetherian and any non-zero \( f \) is contained in only finitely many ideals.

(h) Prove that \( U^{-1}R \) is Noetherian but has infinite Krull dimension.