Worksheet on Normal domains

Let \( R \) be a domain.

**DEFINITION.** The **normalization** of \( R \) is the integral closure of \( R \) in its fraction field. On this worksheet, we let \( \tilde{R} \) denote the normalization of \( R \). A domain \( R \) is **normal** if \( \tilde{R} = R \).

**THEOREM 1.** A one dimensional Noetherian local ring is a DVR.

**THEOREM 2.** Let \( R \) be Noetherian and normal, \( f \in R \setminus \{0\} \). Every associated prime of \( \langle f \rangle \) has height one.

**DEFINITION.** A **divisor** on \( \text{Spec} \ R \) is a formal \( \mathbb{Z} \)-linear combination of height one primes of \( R \). The set of all divisors on \( \text{Spec} \ R \) form a free abelian group \( \text{Div}(R) \).

**DEFINITION.** Let \( R \) be a normal domain. For non-zero \( f \in R \) define the **divisor** of \( f \) to be the formal sum \( \sum_{i=1}^{t} a_i [P_i] \) where \( P_1, \ldots, P_t \) are the minimal primes of \( \langle f \rangle \) and \( a_i = \nu_{P_i}(f) \) where \( \nu_{P_i} \) is the valuation of the DVR \( R_{P_i} \).

**DEFINITION.** The **divisor class group**\(^1\) of \( \text{Spec} \ R \), denoted \( Cl(R) \) is the group \( \text{Div}(R) \) modulo the subgroup generated by divisors of non-zero elements \( f \in R \).

1. **Normalization commutes with localization.** Let \( R \) be a domain and \( U \subset R \) any multiplicative subset.
   (a) Be sure you understand why \( R \) and \( U^{-1}R \) have the same fraction field. Call it \( K \).
   (b) Prove that if \( \frac{a}{b} \in K \) is integral over \( R \), then \( \frac{a}{b} \) is integral over \( U^{-1}R \).
   (c) Conclude that \( \tilde{R} \subset U^{-1}R \) and also \( U^{-1}\tilde{R} \subset U^{-1}R \) as subsets of \( K \).
   (d) Show that if \( \frac{a}{b} \in K \) satisfies an equation of integral dependence
   \[
   X^n + \frac{r_1}{u_1} X^{n-1} + \cdots + \frac{r_{n-1}}{u_{n-1}} X + \frac{r_n}{u_n} \in U^{-1}R[X]
   \]
   over \( U^{-1}R \), then \( \frac{ua}{b} \in K \) is integral over \( R \) where \( u = \Pi_{i=1}^{n} u_i \).
   (e) Prove that \( U^{-1}\tilde{R} = U^{-1}R \). That is, *normalization commutes with localization.*

2. **Normality is a local property.** Prove the following are equivalent for a domain \( R \).
   (i) \( R \) is normal.
   (ii) \( U^{-1}R \) is normal for all multiplicative sets \( U \subset R \).
   (iii) \( R_P \) is normal for all \( P \in \text{Spec} \ R \).
   (iv) \( R_m \) is normal for all \( m \in \text{maxSpec} \ R \).
   [Hint: For (iv) implies (i), check the triviality of the \( R \)-module \( \tilde{R}/R \) locally.]

3. Prove that a UFD is normal.

\(^1\)Also called the **ideal class group** in number theory, particularly for number rings (finite extensions of \( \mathbb{Z} \)).
(4) Prove that an intersection of normal rings (with fraction field \( K \)) is normal. Use Theorem 1 to deduce that if \( R \) is Noetherian and normal, then the ring \( \bigcap_{ht1P \in SpecR} R_P \) is normal. [In fact: for Noetherian rings, \( R \) is normal if and only if \( R = \bigcap_{ht1P \in SpecR} R_P \); See Hochster’s Dec 4 lecture.]

(5) Let \( R \) be a Noetherian normal domain with fraction field \( K \). Define a group homomorphism \( K^* \to \text{Div} R \) so that the divisor class group of \( R \) is the cokernel.

(6) Let \( R \) be a normal Noetherian domain. Let \( q \) be \( p \)-primary where \( p \) is height one.
   (a) Show that \( q = p^{(n)} \) for some \( n \), where by definition, \( p^{(n)} = p^n R_p \cap R \).
   [Hint: Recall that \( q \) is \( p \)-primary if and only if \( \sqrt{q} = p \) and \( q R_p \cap R = q \). You will also need Thm 1.]
   (b) Show \( \text{div}(f) = n_1[p_1] + \cdots + n_t[p_t] \) where \( \langle f \rangle = p_1^{(n_1)} \cap \cdots \cap p_t^{(n_t)} \) is a primary decomp.

(7) **Dedekind Domains.** A **Dedekind domain** is a normal Noetherian ring of dimension 1. Show that the normalization of any finite integral extension of \( \mathbb{Z} \) is a Dedekind domain. Such a ring is called a **number ring**. [You may assume that the normalization of a finitely generated \( \mathbb{Z} \)-algebra is Noetherian. This is a non-trivial fact.]

(8) Let \( R \) be a Dedekind domain. Let \( q \) and \( q' \) be non-zero primary ideals with distinct radicals.
   (a) Show \( q \cap q' = qq' \). [Hint: Observe \( q + q' = R \).]
   (b) Explain why the primary decomposition of any ideal in \( R \) is unique.
   (c) Show \( p^n \neq p^{n+1} \) for all \( n \in \mathbb{N} \) and all non-zero \( p \in \text{Spec} R \). [Hint: NAK!]
   (d) Let \( q \) be \( p \)-primary. Show that \( q = p^n \) for some \( n \). [Hint: Recall that \( q \) is \( p \)-primary if and only if \( \sqrt{q} = p \) and \( q R_p \cap R = q \). You will also need Theorem 1.]
   (e) Prove ideals in a Dedekind domain factor uniquely as a product of prime ideals.

(9) Let \( R = K[x, y, z, w]/\langle xy - zw \rangle \).
   (a) Prove that \( R \) is a three dimensional domain.
   (b) Prove that \( R \) is not a UFD.
   (c) Prove that \( \langle x, z \rangle, \langle x, w \rangle, \langle y, z \rangle, \langle y, w \rangle \) are all height one prime ideals.
   (d) Show that \( R_{(x, z)} \cong K[z, y, \frac{1}{w}]_{(z)} \cong K[z, y, w]_{(z)} \).
   (e) Compute the divisor of \( \pi \in R \). [Hint: First find its minimal primes.]
   (f) Show that all primes in (c) are equal in \( \text{Cl}(R) \) up to sign.
   (g) * Prove that \( \text{Cl}(R) \) is isomorphic to \( \mathbb{Z} \).

(10) Let \( V \) be a valuation ring with fraction field \( V \). Prove that \( V \) is normal. [Hint: For \( \lambda \in K \), consider an equation of integral dependence over \( V \) and the possible values of \( \nu \) for its terms.]

(11) **Polynomial Rings over normal rings are Normal.**
   (a) Let \( R \) be a normal domain. Prove that \( R[x] \) is a Normal domain. [Hint: Use the fact that \( K[x] \) is a UFD to reduce to considering elements of \( K[x] \) integral over \( R[x] \).]
   (b) Prove that a directed union of normal domains is normal.
   (c) Prove that a polynomial ring over any normal ring in any number (even infinite) of variables is normal.

(12) * Prove Theorem 1. [The proof of Theorem 2 is in Mel’s notes on Dec 4.]

---

2See the paper “Noetherian rings without finite normalization” by Olberding for a detailed discussion.