217 Orthonormality and Projections
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Definition: Fix a subspace V of $\mathbb{R}^n$. The orthogonal complement $V^\perp$ to V is the subspace

$$V^\perp = \{\vec{w} \in \mathbb{R}^n \mid \vec{w} \cdot v = 0 \;\; for\ all\ v \in V\}.$$

Theorem: Convenient Formula for Orthogonal Projection.
Fix a subspace V of $\mathbb{R}^n$, and let $\mathbb{R}^n \xrightarrow{\pi_V} \mathbb{R}^n$ be the orthogonal projection onto V. Then for any vector $\vec{x} \in \mathbb{R}^n$, we have

$$\pi_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \cdots + (\vec{x} \cdot \vec{u}_d)\vec{u}_d$$

where $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d\}$ is any **orthonormal** basis for V.

1. Let L be the subspace of $\mathbb{R}^2$ spanned by $\begin{bmatrix}1\\4\end{bmatrix}$.

a. Draw a picture showing L and $L^\perp$. Find a basis for $L^\perp$.

Solution note: You should see two perpendicular lines, L and $L^\perp$, crossing at the origin. A basis for $L^\perp$ is $[-4 \;\; 1]^T$.

b. Use a *geometric argument* to find the kernel and image of the orthogonal projection π_L. What do these have to do with $L^\perp$ and L?

Solution note: Thinking about the projection geometrically, we see every point is mapped to something in L and the points on L are mapped to themselves. So the image is L. The vectors in the kernel are those that project to the origin: this is exactly $L^\perp$.

c. What is $\dim L^\perp + \dim L$? How can you see this an instance of rank-nullity?

Solution note: $\dim L^\perp + \dim L = 1 + 1 = 2$. It is rank nullity applied to the map π_L, since the kernel of π_2 is $L^\perp$, the image is L, and the source is $\mathbb{R}^2$.

d. Find a formula for the projection $\pi_L\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$. [Use the Theorem!]

Solution note: Using the theorem, we need an orthonormal basis for the space L to which we are projecting. We can take $\vec{u}_1 = [1/\sqrt{17} \quad 4\sqrt{17}]^T$. The formula says

$$\pi_L\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = (\begin{bmatrix}x\\y\end{bmatrix} \cdot \vec{u}_1)\vec{u}_1 = (\begin{bmatrix}x\\y\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{17}\\4\sqrt{17}\end{bmatrix})\begin{bmatrix}1/\sqrt{17}\\4\sqrt{17}\end{bmatrix} = \begin{bmatrix}\frac{x+4y}{17}\\\frac{4x+16y}{17}\end{bmatrix}$$

e. Find the matrix A for π_L in standard coordinates.

Solution note: $A = \begin{bmatrix}1/17 & 4/17\\4/17 & 16/17\end{bmatrix}$.

f. Use *geometric thinking* (no formulas!) to find the matrix B for π_L in the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$.

> *Solution note:* Since $\pi_L(\begin{bmatrix} 1 \\ 4 \end{bmatrix}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix})$, the first column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\pi_L(\begin{bmatrix} -4 \\ 1 \end{bmatrix}) = 0$, the second column is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So
>
> $$[\pi_L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

g. Find a matrix S such that $B = S^{-1}AS$.

> *Solution note:* This would be $S_{\mathcal{B} \to \{\vec{e}_1, \vec{e}_2\}}$. This is the 2×2 matrix whose columns are the vectors of $\mathcal{B}$ written in the basis $\{\vec{e}_1, \vec{e}_2\}$. Since $\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\vec{e}_1 + 4\vec{e}_2$, the first column is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Similarly, the second column is $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$. So we can take $S = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$.

2. Let V be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

a. Find a basis for $V^{\perp}$.

> *Solution note:* We are looking for vectors $[a \quad b \quad c]^T$ that have dot product zero with both $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Dotting with the second vector, we see $b = 0$. Dotting with the first, we see $a = -c$. So we get $V^{\perp}$ is the one dimensional subspace with basis $[1 \quad 0 \quad -1]^T$.

b. Find an orthonormal basis for V. Use it to compute the projection of $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the plane V. Now write $\vec{w} = \vec{w}_1 + \vec{w}_2$ where $w_1 \in V$ and $\vec{w}_2 \in V^{\perp}$. Is this expression unique?

Solution note: We can take $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as the required orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ in the theorem. Then the projection is

$$(\vec{w} \cdot \vec{u}_1)\vec{u}_1 + (\vec{w} \cdot \vec{u}_2)\vec{u}_2 = 2\sqrt{2}\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Now we can take this projection to be $\vec{w}_1$, and $\vec{w}_2$ therefore must be $\vec{w} - \vec{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, which is in $V^\perp$. So the decomposition of $\vec{w}$ into its components in V and $V^\perp$ is

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

It is unique.

c. Find the standard matrix of π_V.

Solution note: $A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$.

d. Use your formula to find the kernel of π_V. How does this compare to $V^\perp$?

Solution note: The kernel is the solutions of $A\vec{x} = \vec{0}$. Solving, we see the kernel is the one dimensional space with basis $[1 \;\; 0 \;\; -1]^T$. This is the same as $V^\perp$! This makes sense, since the line through the origin perpendicular to the plane consists exactly of the vectors which map to zero under the projection to the plane.

e. What is $\dim V^\perp + \dim V$? How can you see this an instance of rank-nullity?

Solution note: We see $1 + 2 = 3$. This is rank nullity for π_V, since V is the image of π_V, $V^\perp$ is the kernel of π_V, and $\mathbb{R}^3$ is the source of $\mathbb{R}^3$.

f. Explain why the union of an orthonormal basis for V and one for $V^\perp$ are a basis for $\mathbb{R}^3$. What is the matrix of π_V in this basis?

Solution note: V and $V^\perp$ have only the zero vector in common. So a basis for V and a basis for $V^\perp$ will combine to get a basis for $V + V^\perp = \mathbb{R}^n$. To see why it is an orthonormal basis: we already know all have unit length. We also know that anything in V has dot product against anything in $V^\perp$. And the dot product of two elements both in the orthonormal basis for V or both in the orthonormal basis $V^\perp$ also produces zero.

3. Prove that for any subspace $V \subset \mathbb{R}^n$, $V^\perp$ is a subspace.

Solution note: There are three things to check:

1. $0 \in V^\perp$. This is clear since $0 \cdot \vec{v} = 0$ for any $\vec{v}$.
2. Take $\vec{x}, \vec{y} \in V^\perp$. We need to show $\vec{x} + \vec{y} \in V^\perp$. So we need that $(\vec{x} + \vec{y}) \cdot \vec{v} = 0$ for all $\vec{v} \in V$. Using basic rules of dot product, $(\vec{x} + \vec{y}) \cdot \vec{v} = \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{=}0$, since both $\vec{x}$ and $\vec{y}$ are in $V^\perp$.
3. Take $\vec{x} \in V^\perp$ and scalar c. We need $(c\vec{x}) \cdot \vec{v} = 0$ for all $\vec{v} \in V$. But $(c\vec{x}) \cdot \vec{v} = c(\vec{x} \cdot \vec{v}) = c0 = 0$ using basic properties of dot product and the fact that $\vec{x} \in V^\perp$.

4. Prove that if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}$ is a basis for a subspace $V \subset \mathbb{R}^n$, then

$$V^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \cdots = \vec{x} \cdot \vec{v}_n = 0\}.$$

Solution note: Let W be the set $\{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \cdots = \vec{x} \cdot \vec{v}_n = 0\}$. First we show $V^\perp \subset W$. Take $\vec{y} \in V^\perp$. This means $\vec{y} \cdot \vec{v} = 0$ for all $\vec{v} \in V$. Since the $\vec{v}_i$ are in V, in particular, $\vec{y} \cdot \vec{v}_i = 0$ for all i. This means $\vec{y} \in W$.

We now show $W \subset V^\perp$. Take $\vec{z} \in W$. We need to show $\vec{z} \in V^\perp$. For this, we need $\vec{z} \cdot \vec{v} = 0$ for every $\vec{v} \in V$. An arbitrary $\vec{v}$ in V can be written $a_1\vec{v}_1 + \cdots + a_d\vec{v}_d$, for some scalars a_i, since the $\vec{v}_i$ are a basis for V. So $\vec{z} \cdot \vec{v} = \vec{z} \cdot (a_1\vec{v}_1 + \cdots + a_d\vec{v}_d) = a_1(\vec{z} \cdot \vec{v}_1) + \cdots + a_d(\vec{z} \cdot \vec{v}_d) = 0$, since we know $\vec{z} \cdot \vec{v}_i = 0$ (by definition of $\vec{z} \in W$). So $\vec{z} \in V^\perp$. QED.

5. Prove the following THEOREM: For any subspace V of $\mathbb{R}^n$, $\dim V + \dim V^\perp = n$. [The calculations you did in problems 1 and 2 are a hint!]

Solution note: Consider the map $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ given by projection onto V. The image of this map is clearly V, since everything maps to some vector (the closest vector) in V. It is surjective onto V since each vector in V is sent to itself. The kernel is the set of vectors in $V^\perp$. To see this, we can use the formula for projection onto V given by the Theorem at the top of the worksheet: the formula for $T(\vec{x})$ involves dotting $\vec{x}$ with each element in a basis for V. So we see $T(\vec{x}) = 0$ if and only if $\vec{x} \cdot \vec{v}_i = 0$ for each $\vec{v}_i$ in a basis for V. Hence the kernel of T is $V^\perp$. By rank-nullity, $\dim \ker T + \dim \in T = n$, which is now $\dim V^\perp + \dim V = n$.

Gram-Schmidt Orthonormalization.

6. Let V be the subspace of $\mathbb{R}^5$ spanned by $\begin{bmatrix}1\\9\\9\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$.

   a. Find an orthonormal basis for V.

      [Hint: consider ordering your starting basis carefully to reduce computation time.]

   b. Find the projection of the vector $\begin{bmatrix}1 & 0 & -1 & 0 & 2\end{bmatrix}^T$ onto V.

   c. Find a basis for $V^\perp$. Be clever! Figure out its dimension, then use inspection to find some vectors in $V^\perp$ that *must* be a basis.

   d. Find an orthonormal basis for $\mathbb{R}^5$ that extends your orthonormal basis for V. Find the matrix of the linear transformation π_V in this basis.