Worksheet on Primary Decomposition: Uniqueness

Let $R$ be a commutative ring with 1.

**DEFINITION.** A proper ideal $q$ is **primary** if every zero-divisor in $R/q$ is nilpotent.

**TERMINOLOGY.** A primary ideal $q$ always has prime radical $p$; we say $q$ is **$p$-primary**.

**THEOREM ON UNIQUENESS OF PRIMARY DECOMPOSITION.** Suppose an ideal $J$ in an arbitrary ring admits a primary decomposition

$$J = q_1 \cap q_2 \cap \cdots \cap q_t.$$ 

Then $J$ admits a **minimal primary decomposition**, meaning that the intersection can be assumed irredundant and that the $q_i$ are $p_i$-primary for distinct primes $p_i$. In this case, the set

$$\{\sqrt{q_1}, \sqrt{q_2}, \ldots, \sqrt{q_t}\} = \{p_1, p_2, \ldots, p_t\}$$

is independent of the choice of minimal primary decomposition. Furthermore, the minimal primes among the set $\{p_1, p_2, \ldots, p_t\}$ are precisely the minimal primes of $J$, and for these minimal primes, the corresponding primary component is uniquely determined by $q_i = JR_{p_i} \cap R$.

**NOETHERIAN CASE:** Every ideal in a Noetherian ring admits a minimal primary decomposition $J = q_1 \cap q_2 \cap \cdots \cap q_t$. In this case, the set $\{p_1, p_2, \ldots, p_t\} = \text{Ass}(R/J)$.

**CAUTION:** In the non-Noetherian case, primary decompositions do not always exist for a given $J$, and the radicals of the primary components do not have to be associated primes.

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(1) **LEMMA 1.** Prove that finite intersection commutes with taking radicals, localization, and computing colons. That is, for any finite set of ideals $J_1, \ldots, J_t$ in an arbitrary ring $R$, prove that

(a) $\sqrt{(J_1 \cap J_2 \cdots \cap J_t)} = \sqrt{J_1} \cap \sqrt{J_2} \cdots \cap \sqrt{J_t}$;

(b) $(J_1 \cap J_2 \cdots \cap J_t)U^{-1}R = J_1U^{-1}R \cap J_2U^{-1}R \cdots \cap J_tU^{-1}R$ for any multiplicative set $U \subset R$; and

(c) $(J_1 \cap J_2 \cap \cdots \cap J_t) : x = (J_1 : x) \cap (J_2 : x) \cap \cdots \cap (J_t : x)$ for arbitrary $x \in R$.

(2) **LEMMA 2.** Prove that in any ring, if $P_1 \cap P_2 \cap \cdots \cap P_n$ is an intersection of mutually incomparable prime ideals, then minimal primes of this intersection are precisely the $P_i$. In particular, when is such an intersection prime? [Hint: $P_1P_2\cdots P_n \subset P_1 \cap P_2 \cap \cdots \cap P_n$.]

(3) **MINIMAL PRIMES.** Let $J$ be an ideal in an arbitrary ring which admits a primary decomposition $q_1 \cap q_2 \cdots \cap q_t$.

(a) By grouping together primary ideals with the same radical, explain why we can assume the $q_i$ have distinct radicals $p_i$. [Hint: A finite intersection of $p$-primary ideals is $p$-primary.]

(b) Prove that $\sqrt{J} = \bigcap_{i=1}^t p_i$.

(c) Prove the minimal primes among $\{p_1, p_2, \ldots, p_t\}$ are precisely the min primes of $J$.

(d) Observe that (a) and (c) together establish part of the Theorem on Uniqueness of Primary decomposition.
(4) **Lemma 3.** Let \( q \) be any \( p \)-primary ideal in an arbitrary ring \( R \).
(a) Prove that \( p \subset \sqrt{(q : x)} \). [Hint: First show \( q \subset (q : x) \).]
(b) Prove that
\[
\sqrt{(q : x)} = \begin{cases} \{p, R\} & \text{for } x \notin q \\ \{R\} & \text{for } x \in q. \end{cases}
\]
(c) Prove that for \( x \notin p \), we have \( (q : x) = q \).

(5) **Uniqueness of the \( p_i \).** Let \( J = q_1 \cap q_2 \cdots \cap q_t \) be any primary decomposition of \( J \) in which the radicals \( p_i \) of \( q_i \) are distinct.
(a) Use Lemmas 1 and 3 to show that for any \( x \in R \),
\[
\sqrt{J : x} = \bigcap_{x \notin q_j} p_i.
\]
(b) Fix \( i \). Explain why, if the decomposition is irredundent, we can find \( x \) in every \( q_j \) except \( q_i \).
(c) With \( x \) as in (b), show that \( \sqrt{J : x} = p_i \).
(d) Show that if \( q_1 \cap q_2 \cap \cdots \cap q_t \) and \( q'_1 \cap q'_2 \cap \cdots \cap q'_m \) are two different minimal primary decompositions of an ideal \( J \), then
\[
\{\sqrt{q_1}, \sqrt{q_2}, \ldots, \sqrt{q_t}\} = \{\sqrt{q'_1}, \sqrt{q'_2}, \ldots, \sqrt{q'_m}\}.
\]
In particular \( t = m \). [Hint: Use (a), (c) and Lemma 2.]
(e) Conclude that part of the Theorem on Uniqueness of Primary decomposition is proven.

(6) **Minimal components.** Let \( J = q_1 \cap q_2 \cap \cdots \cap q_t \) be a primary decomposition.
(a) Let \( q \) be \( p \)-primary and \( P \) be any prime ideal such that \( p \not\subset P \). Show that \( qR_P = R_P \).
[Hint: Use Lemma 1(b) and (a).]
(b) Complete the proof of the Uniqueness Theorem for Primary Decomposition.

(7) Let \( R \) be the ring of all sequences of real numbers. Show that \( R \) has infinitely many minimal prime ideals, and therefore the zero ideal does not admit a primary decomposition.

(8) **Lemmas on Associated Primes.** Let \( M \) and \( N \) be arbitrary \( R \)-modules.
(a) Show \( \text{Ass}(M \oplus N) = \text{Ass}(M) \cup \text{Ass}(N) \). [Hint: Consider \( 0 \to M \to M \oplus N \to N \to 0 \).]
(b) Show that \( \text{Ass}(R/J_1 \cap J_2) \subset \text{Ass}(R/J_1) \cup \text{Ass}(R/J_2) \). [Hint: Find \( R/J \to R/J_1 \oplus R/J_2 \).]
(c) For all multiplicative sets \( U \subset R \), we have the following inclusion of sets in \( \text{Spec } U^{-1}R: \)
\[
\{PU^{-1}R \mid P \in \text{Ass}(M) \text{ and } P \cap U = \emptyset\} \subset \text{Ass}(U^{-1}M).
\]
(d) * Prove the converse to (c) when \( R \) is Noetherian.
(e) * For \( R \) Noetherian, show that every minimal prime of \( J \) is in \( \text{Ass}(R/J) \).

(9) Assume that \( R \) is Noetherian. Let \( J = q_1 \cap \cdots \cap q_t \) be a minimal primary decomposition.
(a) Prove that if \( q \) is \( p \)-primary, then \( \text{Ass}(R/q) = \{p\} \). [Hint: If \( Q \in \text{Ass}(R/q) \), then \( Q = (q : x) \) for some \( x \notin q \). Show that \( q \subset Q \subset \sqrt{q} \).]
(b) Show that \( \text{Ass}(R/J) \subset \{\sqrt{q_1}, \ldots, \sqrt{q_t}\} \). Thus all associated primes contribute to the primary components.
(c) * Show that every \( \sqrt{q_i} \) is in \( \text{Ass}(R/J) \). [Hint: Take \( x \in q_i \) for all \( i > 1 \) but not \( q_1 \). Then \( p_1 = \sqrt{J : x} \). So \( p_1 \) is the only minimal prime of \( J : x \) and hence an associated prime of \( J : x \). Note that \( R/p_1 \to R/(J : x) \to R/J \).]