Theorem: Consider a linear transformation $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$. Let $P$ be the parallelepiped which is the image of the standard unit $n$-cube. Then the $n$-volume of $P$ is $|\det T|$.

A. Think about this theorem: What does the determinant of the linear transformation $T$ mean? What is the Theorem saying in the $n = 2$ case? The $n = 3$ case?

B. Let $A$ be a $2 \times 2$ matrix with orthogonal columns, $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$.

1. Explain why the image of the unit square under the transformation $T_A$ (left multiplication by $A$) is a rectangle. Sketch it, and label its sides.

2. Find the area of this image rectangle directly, in terms of $\vec{v}_1$ and $\vec{v}_2$ (not using the Theorem).

3. Verify the theorem for this $A$. [Hint: use the QR factorization of $A$.]

4. Let $B$ be a $3 \times 3$ matrix whose columns $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$ satisfy $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. Verify the theorem for $B$.

Solution note: The first column of the matrix, $\vec{v}_1$, is the image of $\vec{e}_1$, and the second column, $\vec{v}_2$, of $A$ is the image of $\vec{e}_2$. These are two of the the sides of the image parallelogram. Since $\vec{v}_1$ are perpendicular, the image is a rectangle. The sides have lengths $||\vec{v}_1||$ and $||\vec{v}_2||$. So the area is $||\vec{v}_1|| \cdot ||\vec{v}_2||$.

Write $A = QR$, so $|\det A| = |\det Q||\det R|$. The matrix $Q$ has orthonormal columns (and is square) so its determinant is $\pm 1$. The matrix $R$ is the change of basis matrix for the Gram-Schmidt process. Since all we would do is scale each column of $A$ by its length, we know that $R$ is diagonal: $R = \begin{bmatrix} ||\vec{v}_1|| & 0 \\ 0 & ||\vec{v}_2|| \end{bmatrix}$. So the determinant of $R$ is the product $||\vec{v}_1|| \cdot ||\vec{v}_2||$. The exact same proof works for the $3 \times 3$ matrix with orthogonal columns: $B = QR$ where $R = \begin{bmatrix} ||\vec{v}_1|| & 0 & 0 \\ 0 & ||\vec{v}_2|| & 0 \\ 0 & 0 & ||\vec{v}_3|| \end{bmatrix}$.

So $|\det B| = |\det Q||\det R| = ||\vec{v}_1|| \cdot ||\vec{v}_2|| \cdot ||\vec{v}_3||$.

C. Let $A$ be the $2 \times 2$ matrix $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation it defines (by multiplication by $A$).

1. Sketch, in the target $\mathbb{R}^2$, the image of the standard unit square under $T$. If the columns of $A$ are called $\vec{v}_1$ and $\vec{v}_2$, clearly label the vectors $\vec{v}_1$ and $\vec{v}_2$ on your sketch.

2. Suppose we apply the Gram Schmidt process to $\{\vec{v}_1, \vec{v}_2\}$ and get the vectors $\{\vec{u}_1, \vec{u}_2\}$. Add $\vec{u}_1$ to your sketch, clearly showing its relationship to $\vec{v}_1$. Show also $\vec{u}_2$ on your sketch.

3. Find the length of the side (the “base”) of the parallelogram given by $\vec{v}_1$. Explain why this is the same as $\vec{v}_1 \cdot \vec{u}_1$. 


4. Find the height of the parallelogram. Explain why the height is $\vec{v}_2 \cdot \vec{u}_2$.

5. Compute the area of the image parallelogram.

6. Find the $QR$ factorization of $A$.

7. Compute the determinant of $A$ using the $QR$ factorization. Why is it the product of the diagonal elements of $R$? How does it compare to the area in (5)?

Solution note: $\vec{u}_1 = \frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \frac{\vec{v}_2}{||\vec{v}_2||} = \frac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Your sketch should show $\vec{u}_1$ in the same direction as $\vec{v}_1$, whereas $\vec{u}_2$ is perpendicular. The vectors $\vec{v}_1$ and $\vec{v}_2$ are two of the sides of the image parallelogram, and the vector $\vec{v}_2$ is an altitude representing its height. We can think of this vector $\vec{v}_2$ as the component of $\vec{v}_2$ in the $\vec{u}_2$ direction, so its length is $\vec{v}_2 \cdot \vec{u}_2$. So the length of base of our parallelogram is $||\vec{v}_1|| = \vec{v}_1 \cdot \vec{u}_1$ and the height is $||\vec{v}_2|| = \vec{v}_2 \cdot \vec{u}_2$. So the area is "base times height" or $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2) = \sqrt{17} \frac{11}{\sqrt{17}} = \text{The QR factorization is} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{17} & -1/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix} \begin{bmatrix} \sqrt{17} & 7/\sqrt{17} \\ 0 & \frac{11}{\sqrt{17}} \end{bmatrix}$.

So determinant of $A$ is $\det Q \det R = \sqrt{17} \frac{11}{\sqrt{17}} = 11$. 
D. Prove the Theorem for the $2 \times 2$ case using the same technique as in C.

If you need help, here are the steps you should do: Let $A$ be the $2 \times 2$ matrix $[\vec{v}_1 \ \vec{v}_2]$, where $\vec{v}_1$ and $\vec{v}_2$ are two vectors in $\mathbb{R}^2$. Let $T$ be the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by multiplication by $A$.

1. Let $A = QR$ be the QR-factorization of $A$. Write $Q = [\vec{u}_1 \ \vec{u}_2]$. Prove that

$$\det A = (\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2).$$

2. Sketch, in the target $\mathbb{R}^2$, the image of the standard unit square under $T$ and label its vertices.

3. Prove that $(\vec{v}_1 \cdot \vec{u}_1) = ||\vec{v}_1||$ is the length of one of the sides of the image parallelogram (call this side “the base”).

4. Prove that $(\vec{v}_2 \cdot \vec{u}_2)$ is the height of the image parallelogram perpendicular to the base found in (3).

5. Prove the Theorem at the top of the worksheet in the case $n = 2$.

Solution note: We did this above: The image parallelogram has base $\vec{v}_1 \cdot \vec{u}_1$ and the height is $\vec{v}_2 \cdot \vec{u}_2$. The QR factorization is

$$[\vec{v}_1 \ \vec{v}_2] = [\vec{u}_1 \ \vec{u}_2] \begin{bmatrix} \vec{v}_1 \cdot \vec{u}_1 & \vec{v}_2 \cdot \vec{u}_1 \\ 0 & \vec{v}_2 \cdot \vec{u}_2 \end{bmatrix}.$$

So determinant of $A$ is $\det Q \det R = \pm(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2) = \pm\text{height} \times \text{base} = \text{area of image parallelogram}$.

E. The Sign of the Determinant. Let $A$ be a $2 \times 2$ matrix as in (B) representing a linear transformation sending $\vec{e}_1$ to $\vec{v}_1$ and $\vec{e}_2$ to $\vec{v}_2$. Investigate the geometric meaning of the sign of the determinant by sketching $\vec{v}_1$ and $\vec{v}_2$ in several different cases, some where the determinant of $A$ is negative and some where it is positive. What general observation can you make?

Solution note: The sign is positive if the orientation of $\{T(\vec{e}_1), T(\vec{e}_2)\}$ is the same as $\{\vec{v}_1, \vec{v}_2\}$. This means the acute angle between them has $T(\vec{e}_1)$ as the right edge and $T(\vec{e}_2)$ as the left edge (just like the acute angle between $\vec{e}_1$ and $\vec{e}_2$. The sign is negative if the orientation is swapped.

F. Let $A$ be the $3 \times 3$ matrix $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

1. Use the QR-factorization to show that

$$\det A = (\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)(\vec{v}_3 \cdot \vec{u}_3),$$

where $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ is obtained from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ by the Gram-Schmidt process.

2. Imagine the image of the standard unit cube under the linear transformation defined by multiplication by $A$. Why do $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ form three of its edges? Compare to the picture in the book in Section 6.3. What notation does the book have for $(\vec{v}_1 \cdot \vec{u}_1)$, $(\vec{v}_2 \cdot \vec{u}_2)$ and $(\vec{v}_3 \cdot \vec{u}_3)$?

3. The image parallelepiped from (2) has sides that are parallelograms. Explain why one of these sides (let’s call it the “base”) has area $(\vec{v}_1 \cdot \vec{u}_1) \times (\vec{v}_2 \cdot \vec{u}_2)$.

4. Explain why the height of the parallelepiped is $(\vec{v}_3 \cdot \vec{u}_3)$.

5. Show that the volume of the parallelepiped is $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)(\vec{v}_3 \cdot \vec{u}_3)$. 

6. Prove that for a $3 \times 3$ matrix $A$, the volume of the image of the standard unit cube under the linear transformation given by $A$ is $|\det A|$.

7. Do you see why the Theorem on the previous page holds in the case $n = 3$? What about the arbitrary dimensional case?

8. What do you think the sign of the determinant of $A$ tells us?