Problem 1: Borel Groups. Fix a finite dimensional real vector space $V$ of dimension $n$ and a full flag of subspaces:

$$V = V_n \supset V_{n-1} \supset V_{n-2} \supset \ldots \supset V_2 \supset V_1 \supset V_0 = 0$$

where each $V_i$ is a subspace of dimension $i$. Consider the subset $B$ of $GL(V)$ of linear transformations that preserve this flag:

$$B = \{g \in GL(V) \mid gV_i \subset V_i \text{ for all } i\}.$$

1. Show that $B$ is a subgroup of $GL(V)$.

2. By fixing appropriate bases for the $V_i$, show that $B$ can be identified with the subgroup of invertible upper triangular matrices.

3. Prove that $B$ is a Lie group of dimension $\frac{1}{2}(n+1)(n)$.

4. Now fix a partial flag

$$V = V_n \supset V_{d_t} \supset V_{d_t} \supset \ldots \supset V_{d_2} \supset V_{d_1} \supset V_0 = 0$$

where $V_{d_i}$ is a subspace of dimension $d_i$. Let $P \subset GL(V)$ be the subset of linear transformations preserving this flag. What can you say about $P$? Is it also a Lie Group, and if so, can you find its dimension?

Problem 2.

1. Show that conjugation by elements of $SL_2(\mathbb{R})$ induces a four dimensional smooth representation of $SL_2(\mathbb{R})$ on the space of all $2 \times 2$ matrices.

2. Find a one dimensional subrepresentation. What is the action there?

3. Show that the subspace of trace zero matrices forms an irreducible subrepresentation of dimension three.
4. Decompose the four-dimensional space of all matrices (with conjugation action as above) into irreducible representations.

5. Show that the three dimensional component is isomorphic to $V_2$ from Exercise 2. (Hint: index the rows and columns of the matrices by the variables defining the $V_d$).

**Problem 3.** Let $M$ and $N$ be smooth manifolds of dimensions $e$ and $d$.

1. Show that $M \times N$ is a smooth manifold of dimension $d + e$.

2. If $p, q$ are points on $M, N$, show that there is a natural isomorphism $T_{(p,q)}(M \times N) \cong T_pM \times T_qN$.

**Problem 4.** For the group $G = GL_2(\mathbb{R})$, explicitly compute the derivative, at the identity, of the multiplication map. That is, what is the induced map

$$d_e\mu : T_eG \times T_eG \to T_eG$$

where $\mu : G \times G \to G$ is multiplication?

**Problem 5.** Fix a vector space $V$ of dimension $n$ over a field $\mathbb{F}$.

1. Prove that $\Lambda^d V$ is a vector space of dimension $\binom{n}{d}$.

2. Prove that $S^d V$ is a vector space of dimension $\binom{n+d-1}{d-1}$.

3. If a group $G$ acts on $V$ by linear transformations, prove that there is an induced $G$ action also on $\Lambda^d V$ and $S^d V$ by linear transformations.

4. Let $V$ be an irreducible representation of a group $G$ over $\mathbb{R}$ or $\mathbb{C}$. Show that the representation $V \otimes V$ decomposes (as a representation!) as $S^2 V \oplus \Lambda^2 V$.

5. Consider the tautological representation of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ (ie, acting by linear transformations). Explicitly describe the induced one dimensional representation $\Lambda^n \mathbb{R}^n$.

6. Consider the permutation action of $S_3$ on $V = \mathbb{C}^3$. Compute the character of the representations $V \otimes^2, S^2 V$ and $\Lambda^2 V$. Use this to explicitly decompose the nine dimensional representation $V \otimes^2$ into irreducible representations.

**Problem 6: Compact Groups** A topological space is compact if every open cover has a finite subcover. For a subspace $X$ of Euclidean space, $X$ is compact if and only if $X$ is closed and bounded. Which of the following Lie Groups are compact?

$$GL_n(\mathbb{R}), \quad SL_n(\mathbb{R}), \quad S^1, \quad \mathbb{R}^n, \quad \mathbb{R}^* \times \mathbb{R}^*, \quad \mathbb{R}^2/\mathbb{Z}^2, \quad S^1 \times S^1, \quad SO_n, \quad SO(k,l)$$