Problem 1: A study of $S_n$. Let $S_n$ denote the permutation group on $n$ objects.

a. Show that $S_n$ has exactly $n!$ elements.

b. Show that every permutation $\sigma \in S_n$ can be written as a composition of disjoint cycles $\sigma_1 \circ \cdots \circ \sigma_t$ where the $\sigma_i$ are cyclic permutations of some subset of the $n$ objects. Show that this representation is unique, up to reordering the cycles.

c. Show that every permutation is a composition of transpositions (that is, 2-cycles). Are the transpositions unique?

d. Show that there is a way to interpret $D_n$ in a natural way as a subgroup of $S_n$.

e. Find (all) subgroups of $S_n$ isomorphic to $S_k$ for all $k \leq n$.

f. Show that if $k + m \leq n$, then $S_k \times S_m$ is isomorphic to a subgroup of $S_n$. Can you count the number of subgroups of $S_n$ isomorphic to $S_k \times S_m$?

Problem 2: Cyclic Groups. A group is cyclic if it can be generated by a single element.

a. Prove that every cyclic group is abelian.

b. Prove that every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$.

c. Prove that every finite cyclic group is isomorphic to $(\mathbb{Z}_n, +)$, for some $n$.

d. List all cyclic subgroups of $D_4$.

e. How many cyclic subgroups does $D_p$ have, when $p$ is prime?

f. Find a formula for the number of cyclic subgroups of $D_n$, in terms of (the prime factorization of) $n$. 

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Problem 3: Products of Cyclic Groups.

a. Show that \( \mathbb{Z}_4 \) is not isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

b. Show that \( \mathbb{Z}_6 \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_3 \).

c. Can you conjecture a precise condition for when \( \mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n \). 

d. Can you prove it?

Problem 4: Generators and Relations for \( D_n \). Consider the group \( D_n \) of symmetries of the regular \( n \)-gon. Let \( r \) be the counterclockwise rotation though the angle \( \frac{2\pi}{n} \) and let \( s \) be reflection over a line through the center of the \( n \)-gon and any one fixed vertex.

a. Show that \( r \) and \( s \) generate \( D_n \).

b. Show that \( srs = r^{n-1} \).

(c) Show that every element of \( D_n \) can be written uniquely in the form \( s^k r^i \) where \( k = 0 \) or \( 1 \) and \( i = 0, \ldots, n - 1 \).

(d) Is any group generated by two elements \( x \) and \( y \), satisfying \( x^2 = e, y^n = e \) and \( xy = y^{-1}x \) is isomorphic to \( D_n \)?

Problem 5: The order of subgroups.

a. Describe all subgroups of \( D_{12} \). Note their orders, in relation to the order of \( D_{12} \).

b. Make a conjecture about the orders of subgroups of a fixed group. If you already know the theorem, try to prove it. (We will state and prove such a theorem in class eventually).

Problem 6: Classification of Small Order Groups.

a. Show that every group of order three or less is isomorphic to \( \mathbb{Z}_n \).

b. Show that every group of order four is abelian.

c. Show that there are, up to isomorphism, exactly two groups of order four.

d. Show that there is, up to isomorphism, exactly one group of order five.