Worksheet on Reviewing Tensor Product

Let $R$ be a commutative ring with 1. Let $R$-Mod denote the category of $R$-modules.

**Universal Property of Tensor Products:** Any bilinear $R$-module map $M \times N \to Q$ factors uniquely through the universal bilinear map $M \times N \to M \otimes_R N$ sending $(m, n) \mapsto m \otimes n$.

**Properties of Tensor.** Let $A, B, C$ be $R$-modules.

1. **Identity:** $R \otimes_R A \cong A$ and $A \otimes_R R \cong A$.
2. **Associativity:** $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$.
3. **Distributivity:** $A \otimes_R (B \oplus_R C) \cong (A \otimes_R B) \oplus_R (A \otimes_R C)$.
4. **Commutativity:** $A \otimes_R B \cong B \otimes_R A$. **This one is hard! We won’t use it.**

**Definition.** A covariant 1 functor $\Gamma$ from $R$-\text{Mod} to $S$-\text{Mod} is right exact if it preserves cokernels—that is, if for every exact sequence of $R$-modules

$$M_1 \to M_2 \to M_3 \to 0,$$

the sequence

$$\Gamma(M_1) \to \Gamma(M_2) \to \Gamma(M_3) \to 0.$$

Similarly, it is left exact if it preserves kernels (meaning that if $0 \to M_1 \to M_2 \to M_3$ is exact, then so is $0 \to \Gamma(M_1) \to \Gamma(M_2) \to \Gamma(M_3)$.) The functor $\Gamma$ is exact if it is both left and right exact.

**Definition:** An $R$-module $M$ is flat if the functor $N \mapsto M \otimes_R N$ from $R$-\text{mod} to $R$-\text{mod} is exact.

---

1 Tensor Products of Vector Spaces. Let $M$ and $N$ be finite dimensional vector spaces over a field $K$ of dimensions $m, n$ respectively.

(a) Fix bases so that $M \cong K^m$ and $N \cong K^n$ are identified with spaces of column vectors in the usual way. Using the matrix multiplication map

$$K^m \times K^n \to K^{m \times n} \quad (\vec{v}, \vec{w}) \mapsto \vec{v} \cdot \vec{w}^T,$$

prove that $K^m \otimes_K K^n$ is naturally 2 isomorphic to the space $K^{m \times n}$ of $m \times n$ matrices.

(b) Conclude that $M \otimes_K N$ has dimension $mn$, and describe an explicit basis in terms of bases for $M$ and $N$.

(c) Let $X \subset K^{m \times n}$ be the image of the bilinear map in (a). Explain why $X$ consists of the matrices of rank at most 1, and why this is an algebraic set.

(d) How likely is it that a randomly chosen element of $M \otimes_K N$ can be written as $m \otimes n$?

2 Explain why the tensor product of finitely generated modules is finitely generated, and why the tensor product of free modules (of rank $m$ and $n$ respectively) is free (of rank $mn$).

3 Right Exactness of Tensor. For fixed $M$, consider the covariant functor $M \otimes_R -$ from $R$-mod to $R$-mod given by $N \mapsto M \otimes_R N$.

---

1 Left and right exactness are defined similarly for contravariant functors.

2 meaning that the isomorphism can be described without choosing a basis.
(a) Prove that $M \otimes_R -$ is right exact. [Hint: Find a surjection $M \otimes_R M_2/ \text{im}(M \otimes_R M_1) \to M \otimes_R M_3$. To show it is an isomorphism, construct a bilinear map $M \times M_3 \to M \otimes_R M_2/ \text{im}(M \otimes_R M_1)$. Be sure to check your bilinear map is well-defined.]

(b) Prove that $M \otimes_R -$ is not left exact in general. [Hint: Take $M = R/I$ and consider a sequence $0 \to R \xrightarrow{f} R \to R/(f)$ where $f \in I$ is a non-zero-divisor of $R$.]

(c) Prove free modules are flat. Conclude that $V \otimes_K -$ is exact in the category $K$-Vector spaces, for any $V$.

(4) Localization. Let $U \subset R$ be a multiplicative system. For an $R$-module $M$, define $U^{-1}M$ as the set of equivalence classes $\frac{m}{u}$ where $m \in M$ and $u \in U$ with $\frac{m}{u} \sim \frac{m'}{u'}$ if there exists $v \in U$ such that $v(u'm - um') = 0$.

(a) Explain why $M \mapsto U^{-1}M$ is a functor from $R$-mod to $U^{-1}R$-mod.

(b) For $R = \mathbb{Z}$ and $U = \mathbb{Z} \setminus \{0\}$, show that the localization functor kills torsion modules.

(c) Prove there is a unique isomorphism $U^{-1}R \otimes_R M \cong U^{-1}M$. [Hint: Universal properties.]

(d) Prove that the localization functor is exact.

(e) Show that $U^{-1}R$ is a flat $R$-module.

(5) Base Change. Let $R \to S$ be a ring homomorphism.

(a) Show that $M \mapsto S \otimes_R M$ is a right exact functor from $R$-mod to $S$-mod.

(b) In particular, show that $S \otimes_R R/I \cong S/IS$ for all ideals $I \subset R$ and $S \otimes_R R[x_1, \ldots, x_n]/J \cong S[x_1, \ldots, x_n]/JS[x_1, \ldots, x_n]$ for all ideals $J \subset R[x_1, \ldots, x_n]$.

(c) Show that this functor is not left exact in general. [Hint: Use (5).]

(d) For any multiplicative system $U \subset R$, show that the functor from $R$-mod to $U^{-1}R$-mod sending $M \mapsto U^{-1}R \otimes_R M$ is exact.

(e) Show that if $S$ is a flat $R$-algebra, then $S \otimes_R I \cong IS$ for all ideals $I \subset R$.

(6) Adjointness of Tensor and Hom. Let $R \to S$ be a ring homomorphism. Let $M$ and $N$ be $S$ modules, and let $Q$ be an $R$-module.

(a) Discuss natural $R$-module structures on $M$ and $N$.

(b) Discuss a natural $S$-module structure on $\text{Hom}_R(N, Q)$.

(c) Given a $R$-bilinear map $M \times N \to Q$, describe a natural $R$-module map $M \to \text{Hom}_R(N, Q)$.

(d) Show that there is an $R$-module isomorphism $	ext{Hom}_R(M \otimes_R N, Q) \cong \text{Hom}_R(M, \text{Hom}_R(N, Q))$.

(e) Show that there is an $S$-module isomorphism $	ext{Hom}_R(M \otimes_S N, Q) \cong \text{Hom}_S(M, \text{Hom}_R(N, Q))$.

(7) Tensor Product of Algebras. Let $A$ and $B$ be $R$-algebras.

(a) Show that $A \otimes_R B$ has the structure of an $R$-algebra.

(b) Show that there are $R$-algebra maps $A \to A \otimes_R B$ and $B \to A \otimes_R B$ which make $A \otimes_R B$ into a coproduct in the category of $R$-algebras. [Meaning: given any $R$-algebra $T$ to which both $A$ and $B$ map, $\exists$ $R$-algebra map $A \otimes_R B \to T$ making the relevant diagrams commute.]

(8) Let $K$ be an algebraically closed field, and let $V \subset K^n$ and $W \subset K^m$ be algebraic sets.

(a) Show that $V \times W$ is an algebraic set in $K^{n+m}$.

(b) * Show that its coordinate ring is isomorphic to $K[V] \otimes_K K[W]$. 