Math 217: GEOMETRY IN THREE SPACE

Theorem: *A composition of two rotations in* $\mathbb{R}^3$ *(around lines through the origin) is another rotation (around some third line through the origin).*

The purpose of this worksheet is to prove this non-trivial theorem.

WARM-UP: ROTATIONS IN THE PLANE.

1. Is the composition of two rotations (around the origin) a rotation in $\mathbb{R}^2$? If ρ_1 is rotation through θ_1 and ρ_2 is rotation through θ_2, describe the composite explicitly.
2. Is the map given by multiplication by $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ a rotation? Through which angle?
3. Show that a 2×2 matrix A represents a rotation in the plane (in standard coordinates) if and only if A is orthogonal with determinant 1. [Hint: given one column of A, find the other explicitly. Now find the desired θ.]

ROTATIONS IN 3-SPACE.

A. In $\mathbb{R}^3$, a rotation has some *axis*— a line L through the origin— around which the rotation takes place. Think about two rotations in $\mathbb{R}^3$, called ρ_1 and ρ_2, around L_1 and L_2 respectively, and through angles θ_1 and θ_2 respectively. Discuss with your table why the argument that a composition of rotations is a rotation in dimension two does not immediately generalize to 3D. Can you find a special case where it does?

B. Let T_1 be rotation by $\pi/2$ counterclockwise around the z-axis and let T_2 be rotation around the x-axis by $\pi/2$ (counterclockwise).

1. Find the (standard) matrix of the composition $T_1 \circ T_2$.
2. Find the eigenvalues and the algebraic and geometric multiplicity of each eigenvalue.
3. Find a basis for the eigenspace of each eigenvalue.
4. According to the Theorem, we know that $T_1 \circ T_2$ is some rotation. What is the axis of this rotation? [Hint: Think geometrically—what does this axis have to do with eigenvectors?] Find an explicit spanning vector.

C. Explain why rotation ρ (around L, through θ) in $\mathbb{R}^3$ is an orthogonal transformation. Discuss with your table how to find a convenient basis for understanding it. Prove that the determinant of ρ is 1. [Hint: find the matrix in a well-chosen orthonormal basis.]

D. Prove that the composition T of two rotations (around possibly different axes!) in $\mathbb{R}^3$ is orthogonal of determinant one.

E. Show that the only possible eigenvalues of an orthogonal transformation are 1 and -1. [This is easy from the definitions].

F. Show that 1 is an eigenvalue of the composition of two rotations T. [Hint: compute $\det(A - I_3)$ using the fact that orthogonal matrices satisfy $A^T A = I_n$.[1]]

Let T be the composition of two rotations in $\mathbb{R}^3$.
Let L be the line spanned by some eigenvector $\vec{w}$ with eigenvalue 1.
Our goal is to show that T is rotation around L.
Imagine geometrically and think about why this makes sense.

G. Let $W = L^\perp$. Prove that $T(\vec{v}) \in W$ for all $\vec{v} \in W$. Discuss with your table why this means that the restriction of T to W is an *orthogonal* linear transformation $W \xrightarrow{T|_W} W$.

H. Show that the determinant of $W \xrightarrow{T|_W} W$ is also 1. [Hint: chose a convenient basis for W which extends to a convenient basis for $\mathbb{R}^3$ for analyzing T.]

I. Conclude (using the warmup problems) that $W \xrightarrow{T|_W} W$ is a rotation.

J. Prove the Theorem at the top of the first page.

[1] More hints if you're still stuck: make use of multiplicative property of determinants and some basic facts about transpose.