The Spectral Theorem: An \( n \times n \) matrix is orthogonally diagonalizable if and only if it is symmetric.

Definition: Let \( A \) be an \( n \times n \) matrix. We say that \( A \) is orthogonally diagonalizable if either of the two equivalent conditions holds:

1. There exists an orthogonal matrix \( S \) such that \( S^{-1}AS \) is diagonal;
2. \( A \) has an orthonormal eigenbasis.

A. Consider the linear transformation of \( \mathbb{R}^2 \) given by left multiplication by \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

1. How do I know it has an orthonormal eigenbasis?
2. Describe the map geometrically. Now find an orthogonal eigenbasis geometrically, using pure geometric thinking.
3. Write down an orthogonal \( S \) and diagonal \( D \) such that \( D = S^{-1}AS \). This is called “orthogonally diagonalizing \( A \”).
4. Find a matrix \( U \) such that \( U^T AU \) is diagonal.

B. Let \( A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \).

1. Without computing anything, how do I know \( A \) is diagonalizable? What can you say about its eigenvectors?
2. Find the eigenvalues of \( A \).
3. Find one eigenvector of \( A \).
4. Use the spectral theorem to find another eigenvector of \( A \).
5. Find an orthonormal eigenbasis for the transformation \( T_A \).
6. Find orthogonal \( S \) such that \( S^{-1}AS \) is diagonal.
7. Is there an \( S \) such that \( S^TAS \) is diagonal?
8. I claim that there is a unit square in \( \mathbb{R}^2 \) such that \( T_A \) takes that square to a rectangle whose sides are length 2 and 4. Explain.

C. Define symmetric matrix. Prove that if \( A \) is orthogonally diagonalizable, then \( A \) is symmetric. [This proves one direction of the spectral theorem.]

D. Find an example of a matrix that is diagonalizable but not orthogonally diagonalizable.
E. **Prove or Disprove**: Justify or give a counterexample:

1. If $A$ is symmetric, then there is a matrix $S$ such that $S^TAS$ is diagonal.
2. Every orthogonal matrix is orthogonally diagonalizable.
3. If $A$ has an orthonormal eigenbasis, then every eigenbasis is orthonormal.
4. If $P$ is any $5 \times 9$ matrix, then $PP^T$ has an orthonormal eigenbasis.

F. Fix a matrix $A \neq kI_n$ for any scalar $k$. Consider the linear transformation $\mathbb{R}^{n\times n} \xrightarrow{f_A} \mathbb{R}^{n\times n}$ sending a matrix $X$ to $AX - XA$.

1. Show that $I_3$ is an eigenvector of $f_A$. Show that every matrix of the form $A^t$, where $t$ is a natural number, is an eigenvector.
2. Prove that the image of $f_A$ has dimension at most $n^2 - 2$.
3. Find a non-zero, non-identity matrix $A$ such that the zero-eigenspace of $f_A$ is all of $\mathbb{R}^{2\times 2}$.

G. **Prove** that if $A$ is a $2 \times 2$ symmetric matrix, then $f(A) = 0$ where $f$ is the characteristic polynomial of $A$.

H. **The proof of the spectral theorem**.

1. Show that if $A$ can be orthogonally diagonalized, then $A$ is symmetric. [Hint: write $A$ as product of matrices you can easily transpose.]
2. Suppose $A$ is symmetric. Let $\vec{v}_1$ and $\vec{v}_2$ be eigenvectors with *distinct* eigenvalues. Compute the matrix products

   $\vec{v}_1^T A \vec{v}_2$ and $\vec{v}_1^T A^T \vec{v}_2$,

   so that each is expressed in terms of $\vec{v}_1 \cdot \vec{v}_2$.
3. Deduce that if $\lambda$ and $\mu$ are distinct eigenvalues of a symmetric matrix, then the corresponding eigenspaces are orthogonal.
4. Prove that a symmetric matrix is diagonalizable, then it is orthogonally diagonalizable. (Hint: use Gram-Schmidt on each eigenspace).

(This is not a complete proof of the Spectral Theorem—we still need to see why a symmetric matrix is diagonalizable).