Math 412. Supplement on Group Actions

0.1. **Group Actions.** Fix any group \((G, \circ)\).

**Definition:** Let \(X\) be any set. We say the group \(G\) acts on \(X\) if there is a map 
\[ G \times X \to X \quad (g, x) \mapsto g \cdot x, \]

satisfying the following two axioms:
1. \(e_G \cdot x = x\) for all \(x \in X\); and
2. \(h \cdot (g \cdot x) = (h \circ g) \cdot x\) for all \(g, h \in G\) and all \(x \in X\).

Loosely, the group action means that for every element \(g \in G\) and each point \(x \in X\), the element \(g\) “acts on” \(x\) to produce a new point of \(X\), which we denote by \(g \cdot x\). This action should respect the group structure: the element \(e_G\) should do nothing (the first axiom: \(e_G \cdot x = x\)) and acting by \(h\) and then by \(g\) should be the same as acting by \(g \circ h\) (the second axiom: \(h \cdot (g \cdot x) = (h \circ g) \cdot x\)).

**Example 1:** Let \(G\) be the group \(GL_2(\mathbb{R})\) of invertible \(2 \times 2\) matrices, and let \(X\) be the set \(\mathbb{R}^2\). The natural multiplication map defines an action of the group \(GL_2(\mathbb{R})\) on the set \(\mathbb{R}^2\): we define the action of a matrix \(A \in GL_2(\mathbb{R})\) on a vector \(\begin{bmatrix} x \\ y \end{bmatrix}\) by
\[ A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}, \]

the usual matrix product. Each matrix “acts” on each column vector to produce another column vector in a natural way.

The first axiom is satisfied since the identity matrix leaves each column vector unchanged.

The second axiom exactly says that given matrices \(A, B \in GL_2(\mathbb{R})\) and a vector \(\begin{bmatrix} x \\ y \end{bmatrix}\), we have
\[ (AB) \begin{bmatrix} x \\ y \end{bmatrix} = A(B \begin{bmatrix} x \\ y \end{bmatrix}), \]
an instance of the associative law for matrix multiplication!

0.2. **Orbits and Stabilizers.** Fix an action of the group \(G\) on a set \(X\). Consider a point \(x \in X\).

**Definition:** The orbit of \(x\) is the subset of \(X\)
\[ O(x) := \{ g \cdot x \mid g \in G \} \subset X. \]

Loosely, if we think of the action of \(G\) on the set \(X\) as moving the points of \(X\) around, the orbit of a point \(x\) is the collection of all points in \(X\) that we can get to from \(x\) with the action of \(G\).
EXAMPLE 2: Consider the rotation group $SO_2(\mathbb{R}) = \{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \mid \theta \in [0, 2\pi) \}$. It acts on the plane $\mathbb{R}^2$ in an obvious way:
\[
\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta x - \sin \theta y \\ \sin \theta x + \cos \theta y \end{bmatrix}
\]
the rotation of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ through an angle of $\theta$ counter-clockwise.

The identity element of $SO_2(\mathbb{R})$, namely $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ leaves each $\begin{bmatrix} x \\ y \end{bmatrix}$ unchanged, so Axiom 1 is satisfied. Also, rotating through an angle of $\alpha$, and then through $\beta$, is clearly the same as rotating through $\alpha + \beta$, so the second axiom is satisfied.

Let us compute the orbits. First note that if we rotate the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, nothing happens. That is, for any $A \in SO_2(\mathbb{R})$, we have $A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can push the origin to any other point in $\mathbb{R}^2$ with this group action, so
\[
O(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}.
\]

The orbit of the origin is itself. We say that the origin is **fixed point** of this action, since every element of the group leaves it fixed.

Now take any point $p = \begin{bmatrix} a \\ b \end{bmatrix}$ other than $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Applying a rotation $A \in SO_2$, we get another point on the circle with center $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ though $p$. As we apply all the elements of $SO_2$, we get all the points on this circle. That is,
\[
O(\begin{bmatrix} a \\ b \end{bmatrix}) = \text{circle centered at origin through } \begin{bmatrix} a \\ b \end{bmatrix},
\]
which is an infinite set.

**CAUTION:** There can be more than one notation for the same orbit. In fact, there almost always are multiple ways to write an orbit, with no way “preferred” over another. For example, for the action in Example 2 above, the orbit of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the same as the orbit of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: Both $O(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = O(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$ are the same circle of radius one. There are infinitely many ways to write this circle: for any values of $a$ and $b$ such that $a^2 + b^2 = 1$, this same orbit can be written $O(\begin{bmatrix} a \\ b \end{bmatrix})$.

**PROPOSITION:** Suppose that a group $G$ acts on a set $X$. Then the distinct orbits form partition of the set $X$. This means that $X$ is the disjoint union of the distinct orbits: every point of $x$ appears in exactly one orbit. In particular, the cardinalities of the distinct orbits must sum to the cardinality of $X$ (if it is finite).

Let us think about the Proposition in the context of Example 2. The orbits are circles centered at the origin of all possible radii (including the special case of a circle of radius zero, which gives

just the origin). Two such circles are either the same or completely disjoint from each other. Together, their union is all of $\mathbb{R}^2$.

Before proving this, let’s look at some more examples.

**Example 3.** Consider the natural action of $S_5$ on the set of $X = \{1, 2, 3, 4, 5\}$. That is, $\sigma \in S_5$ acts on $i \in \{1, 2, 3, 4, 5\}$ by simply $\sigma(i)$. You should check that this is an action.

What is the orbit of the point $1 \in X$? Well, the transposition $\tau_i = (1\,i)$ sends $1 \mapsto i$ (it also sends $i \mapsto 1$ but that is less relevant at the moment). So basically, we can send 1 to any element of $X$ with this action. This means $O(1) = \{1, 2, 3, 4, 5\}$, the whole set. Likewise, this is also the orbit of 2, or any element of $X$. All the orbits are the same: $O(1) = O(2) = O(3) = O(4) = O(5) = X$. There is only one orbit, and $X$ is the trivial union of its distinct orbit(s).

**Example 4.** Consider the natural action of $S_4$ on the set of $P$ on subsets of $\{1, 2, 3, 4\}$. That is, $\sigma \in S_4$ acts on a subset $Y = \{i_1, i_2, \ldots, i_l\} \subset \{1, 2, 3, 4\}$ by simply $\sigma(Y) = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_l)\} \subset \{1, 2, 3, 4\}$. You should check that this is an action of $S_4$ on $P$.

What is the orbit of the subset $\{1\} \in P$? Well, each $\sigma \in S_4$ will take the set $\{1\}$ to the one element set $\{\sigma(1)\}$, so the orbit contains only one element sets. But also, the transposition $\tau_i = (1\,i)$ sends $\{1\} \mapsto \{i\}$, so the orbit contains all 1-element subsets of $X$. Likewise, the orbit of $\{1, 2\}$ is the set of all two-element sets: assume without loss of generality that $j \neq 1, j \neq 2$, then to get $\{i, j\}$, we apply $(1\,i)(2\,j)$. Thus the orbit is all 2-element subsets of $\{1, 2, 3, 4, 5\}$.

In a similar way, we see that the orbit of $\{1, 2, 3\}$ is the set of all 3-element sets. And the orbit of $\{1, 2, 3, 4\}$ is the set $\{1, 2, 3, 4\}$; that is, $\{1, 2, 3, 4\}$ is a fixed point for this action.

The partition of $P$ into orbits for this action produces five orbits: the set $\{1, 2, 3, 4\}$ and the empty set are both their own orbit. The three remaining orbits consist of the collection of 1-element sets, the collection of 2-element sets, and the collection of 3-element sets.

**Proof of Proposition:** We need to show that every element of $X$ is in some orbit. This is easy: take arbitrary $x \in X$. Its orbit is the set $\{g \cdot x \mid g \in G\}$. Since $e$ is in $G$, the element $e \cdot x = x$ is in $O(x)$. So $x \in O(x)$, which means every element of $x$ is in some orbit.

We next need to check that each element of $X$ is in exactly one orbit. Suppose $z \in O(x) \cap O(y)$. But observe that if $z \in O(x)$, then $z = h \cdot x$ for some $h \in G$. So for all $g \in G$, $g \cdot z = g \cdot (h \cdot x) = (g \circ h) \cdot x$, which means that $O(z) \subset O(x)$. But again, $z = h \cdot x$ implies that $x = h^{-1} \cdot z$, so the same argument shows also $O(x) \subset O(z)$. So $O(z) = O(x)$ and also $O(z) = O(y)$. The orbits $O(x)$ and $O(y)$ are the exact same sets if they are not disjoint. So each element of $X$ can be in at most one orbit. QED

**Caution:** Different orbits can have different sizes. In Example 4 above, the orbit of $\{1\}$ has four elements. The orbit of $\{1, 2, 3, 4\}$ has only one element. This contrasts with the situation for cosets, where all cosets have the same size.

**Definition:** The stabilizer of $x$ is the subset of $G$

$$Stab(x) = \{g \in G \mid g(x) = x\}.$$
**Proposition:** Suppose a group $G$ acts on a set $X$. Then for every $x \in X$, the stabilizer of $x$ is a subgroup of $G$.

**Example 3:** Consider the group $D_4$ of symmetries of the square. It acts on the square in the obvious way. The orbit of a vertex is the entire set of vertices of the square, since each element of $D_4$ will move the vertex to some other vertex, and we can get to any vertex this way. The stabilizer of a vertex are the elements of $D_4$ that fix that vertex. The only symmetries with fix a vertex are the trivial symmetry $e$, and the reflection over the diagonal through that vertex. So the stabilizer of the vertex is either $\{e, a\}$ or $\{e, d\}$, depending on which vertex we have.

**The Orbit-Stabilizer Theorem:** If a finite group $G$ acts on a set $X$, then for every $x \in X$, we have

$$|G| = |O(x)| \times |\text{Stab}(x)|.$$

**Example 3, Continued:** Note that $|D_4|$ has 8 elements. The stabilizer of a vertex, as we saw, has order two. The orbit of a vertex is the full set of 4 vertices, which has cardinality 4. The orbit-stabilizer theorem is confirmed: $8 = 4 \times 2$.

For this same action, let us consider a different point, say, the center of the square. Each symmetry fixes the center, so the orbit consists of the one point, the center, and the stabilizer is all of $D_4$. Again, $8 = 1 \times 8$.

**0.3. Adjunction.** Suppose a group $G$ acts on a set $X$. We have already said that this gives us a way of thinking of every element of $G$ as producing a map $X \to X$. Formally, this can be expressed as follows:

**Theorem:** If a group $G$ acts on a set $X$, there is a group homomorphism

$$G \to \text{Bij}(X) = \{X \to X \mid f \text{ is a bijection.}\}$$

$$g \mapsto [X \to X; \ x \mapsto g \cdot x.]$$

**Important Example:** The **conjugation action** of $G$ on itself is defined by $g \cdot x = gxg^{-1}$. In this case, the bijection induced by $g$

$$G \to G; \quad x \mapsto gxg^{-1}$$

is actually an automorphism of $G$. 