**Vector Spaces and their Transformations.**



## Modelling Vector Spaces on Coordinate Space.

**A. Matrices:** Let $\mathbb{R}^{2\times 2}$ be the vector space of $2\times 2$ matrices.

1. Make sure you can explain exactly why this is a vector space. What is its dimension? Pick an explicit basis $\mathfrak{B}$ for $\mathbb{R}^{2\times 2}$. Chose wisely as you will be writing coordinates in your basis!

   *Solution note:* The dimension is 4. A nice basis is $(E_{11}, E_{12}, E_{21}, E_{22})$, consisting of the $2\times 2$ matrices whose entries are all zero except for a 1 in the subscripted position.

2. Find an example of subspace of $\mathbb{R}^{2\times 2}$ of dimension 1. Find a subspace containing it of dimension 2. Is the set of diagonal matrices a subspace? What about upper triangular matrices?

   *Solution note:* The set of matrices with zeros everywhere except the top left corner is a subspace of dimension 1. The set of diagonal matrices is a subspace containing it of dimension 2. The upper triangular matrices are a subspace of dimension 3.

3. The *trace* of a square matrix is the sum of the diagonal elements. Prove that the map

   $$T : \mathbb{R}^{2\times 2} \to \mathbb{R} \qquad A \mapsto trace(A)$$

   is a linear transformation. Is it surjective? is it injective?

   *Solution note:* $T$ is linear because $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}) = (a+a') + (d+d') = T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) + T(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix})$ and $T(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = T(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}) = ka + kd = k(a+d) = kT(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$. This is surjective, since $T(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = a$ can be any real number. It is not injective, since $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a non-zero element in the kernel.

4. Prove that the set $\mathfrak{sl}_2(\mathbb{R})$ of $2\times 2$ matrices of trace zero is a **subspace** of $\mathbb{R}^{2\times 2}$ and find its dimension.

   *Solution note:* This is the kernel of the trace map. The kernel of any linear transformation is a subspace. Using rank-nullity, we see that the dimension of the image is 1, so the kernel has dimension 3.

5. Find a basis for $\mathfrak{sl}_2(\mathbb{R})$.

   *Solution note:* One basis is $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$.

6. Is the determinant map $\mathbb{R}^{2\times 2} \to \mathbb{R} \quad A \mapsto det(A)$ a linear transformation? Prove your answer.

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| *Solution note:* No! For example, $\det(2I_2) \neq 2\det I_2$. |

7. Using your basis $\mathfrak{B}$ for $\mathbb{R}^{2\times 2}$, find a formula for map

$$\mathbb{R}^{2\times 2} \to \mathbb{R}^4 \qquad A \mapsto [A]_{\mathfrak{B}}.$$

Is this an isomorphism?

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| *Solution note:* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. It is most definitely an isomorphism: it is the coordinate isomorphism! |

8. Now lets look at another basis, say $\mathfrak{A} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$. Make sure you can explain why this is a basis. Find a formula for map

$$\mathbb{R}^{2\times 2} \to \mathbb{R}^4 \qquad A \mapsto [A]_{\mathfrak{A}}.$$

Is this an isomorphism? Which basis do you prefer, $\mathfrak{A}$ or $\mathfrak{B}$. Why?

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| *Solution note:* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a/2 \\ -b \\ (c+d)/2 \\ (c-d)/2 \end{bmatrix}$. It is most definitely an isomorphism: every basis gives a coordinate isomorphism! |

9. Prove that the transpose map

$$T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \qquad A \mapsto A^T$$

is a linear transformation. Is this an isomorphism?

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| *Solution note:* It is easy to check that $(A+B)^T = A^T + B^T$ and $kA^T = (kA)^T$. It is clearly an isomorphism since it is invertible, as it is its own inverse. |

10. Find the matrix of the transpose map $T$ in the basis $\mathfrak{B}$.

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| *Solution note:* $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |

11. Find the matrix of the transpose map $T$ in the basis $\mathfrak{A}$. Is this easier or harder than $\mathfrak{B}$? Why?

*Solution note:* $[T]_{\mathcal{A}} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1/2 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 & 1/2 \end{bmatrix}$

12. Find a matrix $S$ such that $[T]_{\mathfrak{A}} = S^{-1}[T]_{\mathfrak{B}}$ S.

*Solution note:* This is $S_{\mathfrak{A}\to\mathfrak{B}} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

**B. Polynomials:** Let $\mathbb{R}[x]$ be the set of all polynomials (over $\mathbb{R}$).

1. Make sure you can explain why this is a vector space. What is its dimension? Make sure explain why the set $\mathcal{P}_3$ of polynomials of degree 3 or less is a *subspace* of dimension 4. Choose a basis $\mathfrak{B}$ for $\mathcal{P}_3$. Pick a nice simple one as you will be computing with it!

*Solution note:* $(1, x, x^2, x^3)$

2. Is the subset $W$ of $\mathcal{P}_3$ of polynomials divisible by $(x-1)$ a subspace? What about the set of polynomials divisible by $(x-1)^2$? Find at least three linearly independent elements of $W$.

*Solution note:* Yes, these are the polynomials of degree three that can be factored $(x-1)g$. Clearly this set contains 0, is closed under addition (since $(x-1)g_1 + (x-1)g_2 = (x-1)(g_1+g_2)$), and is closed under scalar multiplication (since $k(x-1)g = (x-1)(kg)$.) Similarly, the set of polynomials divisible by $(x-1)^2$ is a subspace. The elements $(x-1), (x-1)x, (x-1)x^2$ are linearly independent elements of $W$.

3. Prove that the "evaluation at 1" map

$$\varepsilon : \mathcal{P}_3 \to \mathbb{R}; \quad f(x) \mapsto f(1)$$

is a linear transformation. Is it surjective? Injective?

*Solution note:* It is surjective but not injective, since the kernel is the subspace of polynomials divisble by $(x-1)$.

4. What is the dimension of the kernel of $\varepsilon$? Can you find a basis for this kernel? [Hint: How does this relate to your subspace in (2)]?

*Solution note:* Its kernel is the space $W$ above. By rank nullity the dimension is 3, so the $((x-1), (x-1)x, (x-1)x^2)$ is a basis.

5. Prove that the map

$$T : \mathcal{P}_3 \to \mathcal{P}_3 \qquad f(x) \mapsto x\frac{d^2 f}{dx^2} - \frac{df}{dx}$$

is a linear transformation.

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| *Solution note:* Since $x\frac{d^2(f+g)}{dx^2} - \frac{d(f+g)}{dx} = x\frac{d^2 f}{dx^2} - \frac{df}{dx} + x\frac{d^2 g}{dx^2} - \frac{dg}{dx}$ and $x\frac{d^2(kf)}{dx^2} - \frac{d(kf)}{dx} = k(x\frac{d^2 f}{dx^2} - \frac{df}{dx})$, the map is linear. |

6. Find the matrix of the linear transformation $T$ on $\mathcal{P}_3$ with respect to your basis $\mathfrak{B}$. Use this $\mathfrak{B}$-matrix to compute the dimensions of the kernel and image of $T$.

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| *Solution note:* The $\mathcal{B}$-matrix is $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The rank is 2, so the image of the map has dimension 2. The kernel has dimension 2 by rank-nullity. |

7. Now use your $\mathfrak{B}$-matrix to find a basis for the kernel and image. Be sure to express your answer again as polynomials (not just coordinates). Confirm by direct computation as well that your purported basis elements are really in the kernel (by applying $T$ and checking you get 0).

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| *Solution note:* The image has basis $(x, x^3)$. The kernel has basis $(1, x^2)$. |

8. Let $\mathfrak{A}$ be the basis $\{x-1, x+1, x^2+2, x^3\}$. Find the $\mathfrak{A}$-matrix of $T$.

9. Explain why $[T]_{\mathfrak{A}}$ and $[T]_{\mathfrak{B}}$ are similar matrices. Find an $S$ that witnesses their similarity. *Hint:* There is a change of basis matrix $S_{\mathfrak{B}\to\mathfrak{A}}$ and a change of basis matrix $S_{\mathfrak{A}\to\mathfrak{B}}$ which might both be relevant. One is *much easier* to write down directly. Why? What is the relation between them?

**C. One reason to chose a non-standard basis:** Consider the map $\mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2$ given by left multiplication by $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

1. How long do you think it would take you to compute $A^{100}$?

2. Let $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = [1 \quad -1]^T$ and $\vec{v}_2 = [1 \quad -2]^T$. Compute $B = [\phi]_{\mathfrak{B}}$.

3. Compute $B^{100}$.

4. Write a matrix equation expressing the relationship between $A$ and $B$.

5. Use (4) to compute $A^{100}$.

6. For what sorts of computations or problems might the basis $\mathfrak{B}$ be superior to the standard? [Note: a basis in which the matrix of $T$ becomes diagonal is called an *eigenbasis* for $T$. Not every transformation has an eigenbasis.]

7. Explain why the transformation "rotation though $\pi/7$" of $\mathbb{R}^2$ can not have an eigenbasis.