Professor Smith Math 295 Lecture Notes

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1 December 2, 2010: The Second Fundamental Theorem and The Cantor Function

1.1 The Second Fundamental Theorem

Theorem: The Second Fundamental Theorem of Calculus: If $f : [a, b] \to \mathbb{R}$ is integrable and there exists a function $g : [a, b] \to \mathbb{R}$ such that $g\prime = f$ then:

$$\int_a^b f = g(b) - g(a) \tag{1}$$

Note: f need not be continuous. An example of this would be $g(x) = x^2 sin(\frac{1}{x})$ which is differentiable on [-1,1] but $g\prime$ is not continuous at 0.

'Cute Proof' of Theorem: The primary tool in proving this theorem will be the Mean Value Theorem which states that if a function g is differentiable on an interval [c,d] then $\exists y \in [c, d]$ $s.t.$ $g\prime(y) = \frac{g(d)-g(c)}{d-c}$.

First, fix a partition P of $[a, b]$ where $P := \{t_0 = a, ..., t_n = b\}$. If we apply MVT to g on $[t_{i-1}, t_i]$ we get:

$$\frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}} = g\prime(y_i) = f(y_i) \tag{2}$$

where $y \in [t_{i-1}, t_i]$. Define

$$m_i = inf\{f(x) \mid x \in [t_{i-1}, t_i]\}, \quad M_i = sup\{f(x) \mid x \in [t_{i-1}, t_i]\}.$$

We know on any interval of the partition P we have $m_i \leq f(y_i) \leq M_i$ by definition of inf and sup. So

$$(t_i - t_{i-1})m_i \leq (t_i - t_{i-1})f(y_i) \leq (t_i - t_{i-1})M_i,$$

since $(t_i - t_{i-1})$ is positive by definition of partition. From line (2), we have $(t_i - t_{i-1})f(y_i) = g(t_i) - g(t_i)$ so this becomes

$$(t_i - t_{i-1})m_i \leq g(t_i) - g(t_{i-1}) \leq (t_i - t_{i-1})M_i.$$

Now if we take the sum over all i the result is quite satisfying:

$$L(f,P) \leq g(t_n) - g(t_0) = g(b) - g(a) \leq U(f,P) \tag{3}$$

Since the center term is a telescoping sum and the left and right when summed are the definition of the upper and lower sums. Since this holds for all partitions we have $sup(L(f,P)) \leq g(b) - g(a) \leq inf(U(f,P))$. Therefore, since f is integrable, we conclude that $g(b) - g(a) = \int_a^b f$. QED

1.2 The Cantor Function

The Cantor Function warns us that there exists a function $f : [0,1] \to [0,1]$ that is continuous, non-decreasing, differentiable *almost* everywhere, with $f(0) = 0,\ f(1) = 1$, and wherever it is differentiable, the derivative is 0. Note: the term *almost everywhere* is a technical term, with a precise mathematical meaning you will learn later.

First, we'll define the Cantor Set. It alone has some peculiar qualities: it is a proper closed subset of $[0,1]$ which is bijective to [0,1] (hence uncountable). But on the other hand it is a *very small* subset of $[0,1]$—so small that it is the complement of a disjoint union of open intervals whose lengths sum to 1! The precise mathematical term for its smallness is that it is "a set of measure zero;" intuitively this means that the probability of selecting an element from the cantor set if one randomly chooses an element from $[0,1]$ is zero. All of these facts about the cantor set can be precisely defined and proved using 295 ideas, and we will eventually get to them.

The Cantor Set C is constructed from $[0,1]$ by successively removing open intervals. First, we remove the middle third open interval, ie $(\frac{1}{3}, \frac{2}{3})$. What remains so far is the disjoint union $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then we remove the middle thirds of each of these two intervals. This leaves us with four closed intervals (each of length $\frac{1}{9}$), and we repeat, removing the open middle third of each of these. Doing this forever produces the Cantor set. Formally, the Cantor set can be written

$$C = [0,1] - \bigcup_{m=1}^{\infty} \left[\bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right). \right]$$

Some of the properties of the cantor set are obvious from its definition. It is closed, being the compliment of an open set (a union of disjoint intervals). It is also clearly "small" in an intuitive sense—if you learned how to do a simple infinite sum in high school (we will do this in 296), you can check that the total length of the removed intervals is 1. In fact, it is not entirely clear what points lie in the cantor set, other than the "endpoints" of the intervals—$\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}$, and so on—which is to say, the rational numbers that can be written with a power of three as the denominator. However, because we claimed that the cantor set is *uncountable,* it must contain more than these endpoints (much more!), since the endpoints are all rational, and hence a countable set. Actually, it is not too hard to see that C is uncountable, using tertiary notation (meaning, writing out real numbers as base three "decimals"). We will do this Friday.

The Cantor function will be defined on $[0,1]$; its range will also be $[0,1]$. (YOU SHOULD DRAW THE PICTURE AS YOU READ THIS!). It looks a lot like a step function—we will describe its values on the "middle third" intervals. The Cantor function $f : [0,1] \to [0,1]$ takes the value $\frac{1}{2}$ on the middle third of $[0,1]$, that is:

$$f(x) = \frac{1}{2} \text{ for all } x \in (\frac{1}{3}, \frac{2}{3}).$$

We need to define the Cantor function on the remaining two thirds $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Looking first at $[0, \frac{1}{3}]$, the Cantor function takes the value $\frac{1}{4}$ on its middle third:

$$f(x) = \frac{1}{2^2} \text{ for all } x \in (\frac{1}{3^2}, \frac{2}{3^2}),$$

and on the middle third of $[\frac{2}{3}, 1]$ it takes the value $\frac{3}{4}$. That is,

$$f(x) = \frac{3}{2^2} \text{ for all } x \in (\frac{7}{3^2}, \frac{8}{3^2}).$$

Now there are still four closed intervals, each of length $\frac{1}{9}$ where we have not yet defined the Cantor function. We define it on the *middle thirds of these* to be $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$, respectively. (**I hope you are drawing the picture!**). For example, we have

$$f(x) = \frac{1}{2^3} \text{ for all } x \in (\frac{1}{3^3}, \frac{2}{3^3}).$$

We continue in this way forever, always defining f on the "middle thirds" of the remaining intervals to be $\frac{k}{2^m}$, for the appropriate values of k and m. If you are drawing the picture (if not, STOP, and do it!), you will see a staircase going up from $(0,0)$ to $(1,1)$ where the size of the some of steps is very small, approaching zero even. The only points at which we haven't defined the Cantor function f yet are the numbers *not* in any of these "middle third" intervals—namely, the points of the Cantor set C. For this, we prove the following lemma:

Lemma: With f defined as above on $[0,1] \setminus C$, there is a unique extension to a continuous function $[0,1] \to [0,1]$.

If you've drawn a nice picture, it should be very believable to you that there is a way to fill in the values of f to be continuous everywhere. You (a hardworking 295 student) might even be able rigorously prove this lemma now; for sure, after a bit more discussion in 296, we will be able to do this easily.

In any case, it should be clear that f is differentiable at any point in one of the "middle thirds," since after all, f is constant on those intervals. Since union of all those middle thirds makes up "almost all" of $[0,1]$, we say that f is differentiable, with derivative zero, "almost everywhere." It is possible to give a precise mathematical meaning to the term "almost everywhere" in such a way to rigorously prove all this. This is routinely done in Math 597 (graduate real analysis), and we will be able to do it this year in Honors Math as well.

To start proving some of the properties about the Cantor set and Cantor function we described above, we would like to have another way to write it down. For this, we use tertiary expansions—meaning base three decimal exansions—which we will review next time.