Exercises for Helsinki Algebraic Geometry Mini-Course

Professor Karen E Smith

2010 Toukokuun 17. 18. ja 19.

Reading: Most of the material of the lectures, can be found in the book *Johdettelua Algebralliseen Geometriaan* by Kahanpää et al, or in *An Invitation to algebraic geometry*, by Smith et al, which is the same book in english.

Two other good (much more comprehensive) books for beginners are Shafarevich’s *Basic Algebraic Geometry, I* (original language: Russian) and Harris’s *Algebraic Geometry: An first course*.

Students who wish to get credit for the course can complete (or attempt) the following problems and send a pdf file (handwritten and scanned is fine) to kesmith@umich.edu. If you are a student at any Finnish University, I will read these and assign a grade any time; for deadlines regarding whether or not credits can be assigned, please ask your professor(s) at your own university; of course, I am happy to contact them regarding the amount of work you completed for the course.

Problem 1: The Zariski Topology.

1. Show that an arbitrary intersection of subvarieties of $\mathbb{C}^n$ is a subvariety of $\mathbb{C}^n$.

2. Show that a finite union of subvarieties of $\mathbb{C}^n$ is a subvariety of $\mathbb{C}^n$.

3. Show that $\mathbb{C}^n$ can be given the structure of a topological space whose closed sets are the subvarieties of $\mathbb{C}^n$. This is called the Zariski Topology on $\mathbb{C}^n$.

4. Show that the Zariski topology on $\mathbb{C}^n$ is strictly coarser than the standard Euclidean topology on $\mathbb{C}^n$. (This means that every Zariski open set is open in the Euclidean topology, but not conversely).

5. Explicitly describe the Zariski open sets of $\mathbb{C}^1$ (without saying “polynomial”).

6. Prove or disprove: Every non-empty Zariski open set in $\mathbb{C}^n$ is dense. Is $\mathbb{C}^n$ Hausdorff?

7. Prove or disprove: The Zariski topology on $\mathbb{C}^n$ is compact (meaning every open cover has a finite subcover).
Discussion: Although in this course we will stick to algebraic geometry over the complex numbers, it is important to realize that the Zariski topology plays a starring role in many branches of mathematics (number theory, arithmetic geometry, representation theory, commutative algebra) where no other topology may be available. For example, if $K$ is any field—say, $\mathbb{Q}$ or a finite field or the field of $p$-adic numbers—then one can define a Zariski topology on $K^n$ exactly as above (with the same proof).

**Problem 2: Ideals and subvarieties.** Let $\{F_i\}_{i \in \Lambda}$ be an arbitrary collection of polynomials in $n$-variables over $\mathbb{C}$, let $I$ be the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ they generate, and let $\text{Rad } I$ be the set of elements polynomials $f$ in $\mathbb{C}[x_1, \ldots, x_n]$ satisfying $f^t \in I$ for some $t$.

1. Show that $\text{Rad } I$ is an ideal.
2. Prove that there is a $T$ such that $f^T \in I$ for all elements $f \in \text{Rad } I$. (Hint: you must use the fact that every ideal in a polynomial ring is finitely generated.)
3. Show that $\text{V}(\{F_i\}_{i \in \Lambda}) = \text{V}(I) = \text{V}(\text{Rad } I)$ in $\mathbb{C}^n$.
4. Show that $\text{I}(\text{V}(I)) = V$ as subsets of $\mathbb{C}^n$.
5. Show that $\text{I}(\text{V}(I)) \subset \text{Rad } I$. The hard part of Hilbert’s Nullstellensatz is that this is actually an equality. In fact, this step uses that the field $\mathbb{C}$ is algebraically closed. Hilbert’s Nullstellensatz fails over non-algebraically closed fields such as $\mathbb{R}$.
6. Find an example to show the failure of the nullstellensatz over $\mathbb{R}$. Hint: Consider the ideal generated by $x^2 + y^2$ in the polynomial ring $\mathbb{R}[x, y]$.

**Discussion:** To a modern arithmetic geometer, the “variety” defined by $x^2 + y^2$ is really a union of two lines through the origin in $\mathbb{C}^2$, or slopes $i$ and $-i$. They each contain only one “real point”, namely their intersection point, the origin. The Galois group of $\mathbb{C}/\mathbb{R}$ (which is the two element group containing only the identity map and complex conjugation) acts on this variety by conjugating the coordinates; this action interchanges the two lines. The fixed points of this action is precisely the real point $(0, 0)$. More generally, a real algebraic variety can be viewed as the fixed points of the conjugation action on a complex variety, generalizing the idea of Galois theory. This is the beginning of the very beautiful subject of arithmetic geometry.

**Problem 3: The coordinate ring.** Fix a variety $V \subset \mathbb{C}^n$. Consider the ring $R$ of all polynomial functions on $\mathbb{C}^n$ restricted to $V$.

1. Find the kernel of the natural restriction map $\mathbb{C}[x_1, \ldots, x_n] \to R$ sending a polynomial to its restriction to $V$.

---

1The proof is not beyond a one-semester course in algebraic geometry or commutative algebra, but it is perhaps unfair to ask you to do in a six hour course.
2. Prove that $R$ has no nilpotent elements.

3. Show that every finitely generated $C$-algebra without nilpotents is the coordinate ring of some variety. (An element $f$ of a ring $R$ is nilpotent if some power is zero.)

4. Use Hilbert’s nullstellensatz to prove that the points of $V$ are in one-one correspondence with the maximal ideals of $R$.

5. Show that $V$ is irreducible as a topological space if and only if the ring $R$ is a domain. [Recall that $V$ is an irreducible topological space if whenever $V = W_1 \cup W_2$ for two closed sets then $V = W_i$ for some $i$. Recall that $R$ is a domain if $fg = 0$ implies either $f$ or $g$ is zero.]

Discussion: The coordinate ring encodes all the algebraic geometry of $V$ in many precise ways; one way to state this is that the category of affine algebraic varieties over $\mathbb{C}$ is equivalent to the category of finitely generated $\mathbb{C}$-algebras without nilpotents; See Chapter 3 of An Invitation. This is the starting point of the deep connection between algebraic geometry and commutative algebra, as well as the gateway to more abstract algebraic geometry (Grothendieck’s theory of schemes) in which the objects of study are geometric objects locally modeled on some commutative rings, which we imagine to be “rings of functions” on spaces which may not exist in reality. Hartshorne’s book Algebraic Geometry is the standard source for beginning scheme theory. Even more recent development include attempts to do “non-commutative algebraic geometry” by imagining the elements of some non-commutative ring to be functions on some (non-existing) space. For more about this ask Kari Vilonen next time he is in Helsinki.

Problem 4. Projective Space and Projective Varieties.

1. A line in $\mathbb{P}^n$ is the set of points in $\mathbb{P}^n$ parametrizing the one dimensional subspaces contained in some two-dimensional subspace of $\mathbb{C}^{n+1}$. Prove that any two lines in $\mathbb{P}^2$ intersect in exactly one point.

2. Prove that complex projective $n$-space $\mathbb{P}^n_\mathbb{C}$ is a compact complex manifold which has an open cover by $n + 1$ charts, each identified with $\mathbb{C}^n$, for which the transition functions are given by rational functions in the coordinates.

3. Show that if $V \subset \mathbb{P}^n$ is a projective variety, then each of the intersections of $V$ with the charts you found in (2) is an affine variety in $\mathbb{C}^n$. Thus, a projective variety is “locally” an affine variety.

4. Show that a “line” $L$ in $\mathbb{P}^2$ as defined in (1) is literally a (complex) line in $\mathbb{C}^2$ after intersecting with any of the three affine charts of $\mathbb{P}^2$.

5. Show that the closure in $\mathbb{P}^2$ of the affine variety $V(xy - 1) \subset \mathbb{C}^2$ (using the embedding of $\mathbb{C}^2$ in $\mathbb{P}^2$ given by $(x, y) \mapsto [x : y : 1]$) is the projective variety defined by the homogeneous equation $XY - Z^2$. 

3
Discussion: The rational functions you (most likely) wrote down for the chart transitions are quotients of polynomials with coefficients of $\pm 1$ only. This means that they make sense over any field. Indeed, the projective $n$-space over $\mathbb{K}$, where $\mathbb{K}$ is any field, is the most basic example an algebraic variety over $\mathbb{K}$ which is not simply a subset of some affine space over $\mathbb{K}$. Of course, just as you showed over $\mathbb{C}$, it is covered by $n + 1$ overlapping (zariiski) open sets, each identified with $\mathbb{K}^n$ and with transitions functions given by regular functions in the coordinates. The object $\mathbb{P}^n_\mathbb{K}$ is a linear algebraic object with a rigid geometry of points, lines, planes, etc.

Problem 5. Moduli Spaces of Hypersurfaces.

1. Show that the set of all hypersurfaces of degree $d$ in $\mathbb{P}^n$ is parametrized by a projective space, and find its dimension. This moduli space is denoted $\mathcal{H}_{d,n}$. (Note that, for example, it is natural to interpret the zero set of $x^d$ as a hypersurface of degree $d$—we interpret this point in the moduli space as geometrically as $d$ copies of hyperplane $x = 0$.)

2. Show that the set of hypersurfaces containing a given point $p \in \mathbb{P}^n$ is parametrized by a hyperplane in the moduli space $\mathcal{H}_{d,n}$.

3. Fix $T$ in $\mathbb{P}^n$. Show that the set of hypersurfaces $\mathcal{X}$ of degree $d$ containing those $T$ points is parametrized by a closed subvariety of $\mathcal{H}_{d,n}$. What is its expected dimension if the $T$ are chosen completely randomly? What is the largest and smallest possible dimension (depending on the set of $T$ points)?

Discussion: The spaces $\mathcal{H}_{d,n}$ are the simplest examples of Hilbert schemes, which are special kinds of moduli spaces parametrizing projective subvarieties of a fixed $\mathbb{P}^n$ which share some special numerical invariants. In the case of $\mathcal{H}_{d,n}$ we parametrized the projective subvarieties of $\mathbb{P}^n$ having degree $d$ and codimension 1. More generally if we fix all the Chern numbers (or coefficients of the so-called Hilbert polynomial), we can similarly parametrize all the subvarieties of $\mathbb{P}^n$. In some cases (for example, for smooth curves), it is also possible to parametrize the set of isomorphism classes of curves (say, of genus $g$ considered as Riemann surfaces), resulting in the moduli space $\mathcal{M}_g$ of curves of genus $g$. This is much harder to do, but very fruitful! Deligne and Mumford found natural compactification, also a variety, whose points have interpretations as degenerate curves. A new wave of recent research in algebraic geometry has been studying different compactifications, trying to interpret the “points at infinity” geometrically.

\footnote{It is not too much harder to show that the set of “degenerate hypersurfaces” forms a closed subvariety of $\mathcal{H}_{d,n}$ of strictly smaller dimension, so the moduli space of non-degenerate hypersurfaces is a dense (Zariski) open set of $\mathcal{H}_{d,n}$.}