Professor Smith Math 295 Lecture Notes

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1 November 1: Compactness

Recall from last class:
Definition: Let S be a subset of a topological space X. We say S is compact if **every** open cover has a *finite* subcover.

1. A finite set is compact in *any* topological space X. Let $S = \{x_1, ..., x_n\}$. Say $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$. Each $x_i \in U_{\lambda_i}$ for some $\lambda_i \in \Lambda$. Then S is also contained in the finite subcover $U_{\lambda_1} \bigcup ... \bigcup U_{\lambda_n}$.

2. If X has the discrete topology, then a subset S of X is compact if *and only if* it is finite. Indeed, since the point-sets $\{x\}$ are all open in the discrete topology, the open cover
$$\bigcup_{x \in S} \{x\}$$
has a finite subcover if and only if S is finite. This shows an infinite set can't be compact (in the discrete topology) , since this particular cover would have no finite cover.

3. **An infinite compact set:** The subset $\bar{S} = \{1/n \mid n \in \mathbb{N}\} \bigcup \{0\}$ in $\mathbb{R}$ is compact (with the Euclidean topology).
Proof that $\bar{S}$ is compact: Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be any open cover of S. Since $0 \in \bar{S}$, we know that there is some open set in our cover, say U_{λ_0}, which contains 0. Because U_{λ_0} is open $\exists \epsilon > 0$ s.t. $B_\epsilon(0) \subset U_{\lambda_0}$.

   By the Archimedean property $\exists n$ such that $1/n < \epsilon$ so $\forall n' > n$ we have $1/n' \in U_{\lambda_0}$. There are only finitely many elements of S then that are not in U_{λ_0}, namely the elements $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n-1}$. Of course, each of these is in some set of the cover, say $\frac{1}{i}$ is in U_{λ_i}. So then we have $S \subseteq U_{\lambda_0} \bigcup U_{\lambda_1} \bigcup ... \bigcup U_{\lambda_{n-1}}$. (Note that U_{λ_0} contains most of the elements of S.) This shows that every open cover has a finite subcover, so S is compact. QED.

4. <u>Example of a non-compact set:</u> The open interval (a, b) is not compact (in the usual Euclidean topology from $\mathbb{R}$). The cover

$$\bigcup_{n \in \mathbb{N}} (a + \frac{1}{n}, b)$$

is an example of an open cover with no finite subcover.

5. <u>Example of a closed set which is non-compact:</u> Take $[b, \infty)$. This is covered by $\bigcup_{n \in \mathbb{N}} (b - 1, b + n)$. But this collection has no finite subcover. If there were, then $S \subset (b - 1, b + n)$ (since the sets are all contained in eachother, which makes any finite union be the largest one). But this is absurd, since $[b, \infty)$ contains, for example, $b + n + 1$, which is not $(b - 1, b + n)$.

1.1 Compactness in $\mathbb{R}$

What does compactness mean in the space we know best? Chris more or less guessed at the following result.

<u>**Theorem 1:**</u> Let $S \subseteq \mathbb{R}$ (with the Euclidean topology). Then S is compact if and only only if S is closed and bounded.

For now, we'll just prove $\Rightarrow$ direction, to build up our intuition of compactness.

<u>**Proof of $\Rightarrow$ bounded:**</u> Assume S is compact. Consider the open cover

$$S \subseteq \bigcup_{n \in \mathbb{N}} (-n, n).$$

Because S is compact, this cover has a finite subcover, which implies $S \subseteq (-N, N)$ for some natural number N. But then S is bounded above by N and below by $-N$.

We now show that a compact set S is closed. We want to show that $\mathbb{R} \setminus S$ is open. Take any $x \in \mathbb{R} \setminus S$ (Since S is bounded its complement is nonempty). Now consider the cover

$$U_\epsilon = (-\infty, x - \epsilon) \bigcup (x + \epsilon, \infty), \quad (1)$$

as ϵ ranges over all positive real numbers. We can say $\bigcup_{\epsilon\in\mathbb{R}_{>\not{\vdash}}} U_\epsilon = \mathbb{R} - \setminus\{x\}$. So then $S \subseteq \bigcup_{\epsilon\in\mathbb{R}_{>\not{\vdash}}} U_\epsilon$ since $x \notin S$.

Because S is compact, this cover has a finite subcover. So

$$S \subseteq U_{\epsilon_1} \bigcup U_{\epsilon_2} \bigcup \cdots \bigcup U_{\epsilon_n} \;=\; U_\epsilon$$

where $\epsilon = min\{\epsilon_1, ..., \epsilon_n\}$. Considering complements, this says that

$$(x - \epsilon, x + \epsilon) \subseteq \mathbb{R} \setminus S.$$

In others words, for any x in the compliment of S, there is a ϵ-neighborhood of x completely contained in the compliment of S. This says that the complement of S is open, so that S is closed. QED.

Example: The closed bounded interval $[a, b]$ is compact, according to this theorem (though we have not yet proven this direction).

Intuitively, a compact set S of $\mathbb{R}$ can not go off to infinity (bounded), nor can it accumulate to some point not in S (closed). It must be both bounded and closed.

1.2 The Extreme Value Theorem

Theorem 2: Let $f : X \to Y$ be a continuous map between topological spaces. If $S \subseteq X$ is compact, then $f(S)$ is compact.

First we will examine a special case of this: the Extreme Value Theorem, which is one of the "hard theorems" from Chapter 7 of Spivak, which says that on a closed bounded interval, a continuous function must *achieve* a minimum and a maximum.

Corollary: If $f : [a, b] \to \mathbb{R}$ is continuous then $\exists x_{min},\ x_{max} \in [a, b]$ s.t. $\forall x \in [a, b]$,

$$f(x_{min}) \leq f(x) \leq f(x_{max}).$$

Proof: Since $[a, b]$ is connected, it is sent to some connected set of $\mathbb{R}$, which is therefore an interval by a Theorem last week. Since $[a, b]$ is compact, Theorem 2 says its image is also compact, hence closed and bounded by Theorem 1. The only kind of closed bounded interval is $[c, d]$. So $f([a, b]) = [c, d]$. This means that c is the minimal value and d is the maximal value that f achieves on the closed interval $[a, b]$. This means there must be $x_{min},\ x_{max} \in [a, b]$ such that $\forall x \in [a, b]$, we have

$$f(x_{min}) \leq f(x) \leq f(x_{max}).$$

QED

Note: This would not work for $(0, 1]$ since we could have a function like $f(x) = 1/x$. It never achieves a maximal value on the half-open interval $(0, 1]$. Similarly, the function $g(x) = x$ never achieves a maximal value on the half open interal $[0, 1)$ even though g is bounded above.

Proof of Theorem 2: Take $\{U_\lambda\}$ to be an open cover of $f(S)$. Note that each $f^{-1}(U_\lambda)$ is open in X by continuity of f, and

$$S \subseteq \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda).$$

So this open cover of S has a finite subcover, which means

$$S \subseteq f^{-1}(U_{\lambda_1}) \bigcup ... \bigcup f^{-1}(U_{\lambda_n}).$$

Applying f gives

$$f(S) \subseteq f(f^{-1}(U_{\lambda_1})) \bigcup ... \bigcup f(f^{-1}(U_{\lambda_n})),$$

which implies that

$$f(S) \subseteq U_{\lambda_1} \bigcup ... \bigcup U_{\lambda_n}.$$

Thus every open cover has a finite subcover, so $f(S)$ is compact. QED