PROFESSOR SMITH MATH 295 LECTURE NOTES

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1. November 2, 2010: More on Compactness.

Assignment: Read the appendix of chapter 8. Also Exam 2 is coming up in just a few weeks, I was planning on November 19. The format will be similar.

ALSO: EXPECT A QUIZ ON THE MATERIAL FROM THE NOTES SOMETIME IN THE NEXT FEW DAYS.

1.1. **Compactness.**

Again: **Definition:** A subset S of a topological space X is compact if every open cover of S has a finite subcover.

Example: The Trivial Topology would make every S compact.

Another topology were all subsets are compact: The Cofinite Topology (also known as the Finite Complement Topology). In this topology the open sets are complements of finite sets in X (as well as $\emptyset$ and X).
Proof it is a topology: take $\{U_\lambda\}_{\lambda\in\Lambda}$ to be open sets in the cofinite topology on X. This means that each $X \setminus U_\lambda$ is finite. But then their union

$$\bigcup_{\lambda\in\Lambda} U_\lambda$$

is also the complement of a finite set. Indeed, let us call the union U. Then U contains each one of the open sets, so U contains one of them, call it U_{λ_1}, which without loss of generality we assume to be non-empty. Say $U_{\lambda_1} = X \setminus \{x_1, \ldots, x_T\}$. Since $X \setminus \{x_1, \ldots, x_T\} \subset U$, we see that after taking complements, $X \setminus U \subset \{x_1, \ldots, x_T\}$. So $X \setminus U$ is also a finite set and U is open in the co-finite topology. So the co-finite open sets are closed under unions. We also have to show that the co-finite open sets are closed under finite intersections. This is also easy—check it yourself! QED

The cofinite topology on $\mathbb{R}$: Here the open sets are complements of finite sets of real numbers (plus the empty set and $\mathbb{R}$ itself). For example, $U_1 = \mathbb{R} - \{1, 2, 3\}$ and $U_2 = \mathbb{R} - \{0, 1, 2\}$ are two open sets. Their union is

$$U_1 \bigcup U_2 = \mathbb{R} \setminus \{1, 2\}$$

and their intersection is

$$U_1 \bigcap U_2 = \mathbb{R} \setminus \{0, 1, 2, 3\}.$$

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Note: Intuitively, if we have a topological space whose open sets are large, then it is easier to find compact sets in the topology. This is because the open sets tend to cover large areas, so there is a better chance a finite collection of them will cover any given set. The following is an example of this phenomenon.

Proposition: If S is any subset of any set X (with the co-finite topology), then S is compact.
Proof: Say $S \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, where the U_λ are open in the cofinite topology. Without loss of generality, assume the U_λ are all non-empty. Each is the complement of some finite set of points in X. Fix any U_{λ_0}, say

$$U_{\lambda_0} = X - \{x_1, ..., x_n\}.$$

Note that U_{λ_0} already covers most of S. Indeed, the only possible points that might not get covered by U_{λ_0} are the points $x_1, \ldots, x_n$—more precisely, the only points of S not already covered by U_λ are exactly the finite set of points in $S \cap \{x_1, ..., x_n\}$. Let us relabel these points, so that this intersection is $\{x_1, \ldots, x_T\}$. Each of these x_i must be in some U_{λ_i} (because they form a cover of S). Finally, we conclude that

$$S \subseteq U_{\lambda_0} \bigcup U_{\lambda_1} \bigcup, \ldots, \bigcup U_{\lambda_T}.$$

Thus the arbitrary cover of S has a finite subcover, showing that S is compact. QED.

1.2. **Proof of Generalized Extreme Value Theorem.** Last time, we proved that continuous maps take compact sets to compact sets, which we interpreted as a generalized form of the Extreme Value Theorem. In order to specialize this to the Euclidean topology on $\mathbb{R}$ and deduce the Extreme Value Theorem, we needed the following theorem:

Theorem 1.1. *A subset S of the real line is compact (in the Euclidean topology) if and only if S is closed and bounded.*

Proof. We proved yesterday that compact implies closed and bounded. It remains only to prove the other direction, that a closed bounded set is always compact. The hardest part is the following lemma.

Lemma 1.2. *The closed bounded interval $[a, b]$ is compact.*

Proof of Lemma. Let $\mathcal{C} := \{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of [a,b]. We need to show it has a finite subcover.

Let

$$A = \{x \in [a, b] \mid [a, x] \text{ is covered by some finite subcover of } \mathcal{C}\}.$$

The set A is not empty, since $a \in A$ (the degenerate interval $[a, a] = \{a\}$ is covered by at least one of the open sets in the cover). The set A is also bounded above by b. Now, by the Completeness Axiom for $\mathbb{R}$, we know that A has a supremum because it is nonempty and bounded. Call it α.

Since $\alpha \in [a, b]$, we know that α must be in at least one of the open sets, say U_{λ_0}, of the cover. By definition of openness, we can fix $\delta > 0$ such that

$$(\alpha - \delta, \alpha + \delta) \subseteq U_{\lambda_0}.$$

By shrinking δ if necessary, we can assume also that $a < \alpha - \delta$ (unless $a = \alpha$) and that $\alpha + \delta < b$ (unless $\alpha = b$).

Claim: The supremum α is in A. To see this, note that we are done if $\alpha = a$, so assume $a < \alpha$. Take any $y \in (\alpha - \delta, \alpha)$. Since $y < \alpha$, we have that $y \in A$ and so $[a, y]$ is covered by some finite subcover of $\mathcal{C}$. That is,

$$[a, y] \subset U_{\lambda_1} \bigcup \cdots \bigcup U_{\lambda_t}$$

where the U_{λ_i} here are sets in the collection $\mathcal{C}$. But then

$$[a, \alpha] \subset U_{\lambda_0} \bigcup U_{\lambda_1} \bigcup \cdots \bigcup U_{\lambda_t}$$

which shows that $\alpha \in A$, proving the claim.

Finally, we note that the supremum is b. Indeed, if $\alpha = b$, we are done, so we assume $\alpha < b$. In this case we choose some $y' \in [a, b]$ slightly larger than α, say in the interval $(\alpha, \alpha + \delta)$. Clearly we have

$$[a, y'] \subset U_{\lambda_0} \bigcup U_{\lambda_1} \bigcup \cdots \bigcup U_{\lambda_t}$$

as well, since the δ-neighborhood of α is entirely contained in U_{λ_0}. This shows that y is also in A and so α could not have been the supremum of A, since it is not even an upper bound. This contradiction forces $\alpha = b$, so we conclude that the closed interval $[a, b]$ is covered by some finite union of sets from $\mathcal{C}$. This shows that $[a, b]$ is compact, completing the proof of the Lemma. □

Now, we complete the proof of the Theorem. Take any $S \subseteq \mathbb{R}$ that is closed and bounded. Now take any open cover $\mathcal{D} := \{U_\lambda\}_{\lambda \in \Lambda}$. Because S is bounded, $S \subseteq$ [-N,N] for some $N \in \mathbb{N}$. We also know [-N,N] is compact by the lemma.

Now consider the cover of $\mathbb{R}$ obtained by adding the one open set $U = \mathbb{R} \setminus S$ to the cover $\mathcal{D}$. Since $[-N, N]$ is also covered by this cover, and it is compact, we must have that $[-N, N]$ is covered by some finite subcollection. That is:

$$[-N, N] \subset U_{\lambda_1} \bigcup \cdots \bigcup U_{\lambda_t} \bigcup U$$

where $U = \mathbb{R} \setminus S$. But then since $S \subset [-N, N]$, we have

$$S \subset U_{\lambda_1} \bigcup \cdots \bigcup U_{\lambda_t} \bigcup U. \tag{1}$$

The set $U = \mathbb{R} \setminus S$ is not in the original cover $\mathcal{D}$, so this does not yet give a finite subcover of our original cover. However, note that the set U is completely useless for covering S. If the inclusion in line 1 holds, then also we must have

$$S \subset U_{\lambda_1} \bigcup \cdots \bigcup U_{\lambda_t},$$

since $S \bigcap U = \emptyset$. This shows that the original cover has a finite subcover, so S is compact.

□

Using the exact same idea as in the last paragraph of the proof, you can prove the following general theorem.

Corollary 1.3. *Closed subsets contained in compact sets are compact, in any topological space X.*

You should make sure you can prove this statement, as you may see it again.