PROFESSOR SMITH'S MATH 295 LECTURE NOTES

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Note: There is a typo on problem 3c. The words "left" and "right" are switched. Also, don't forget to read Chapter 8 appendix and be ready for a quiz!

1. Summary of Topology Discussion

Over the past 2 weeks, we have proved three types of theorems: general topological theorems, theorems which interpret topological properties in $\mathbb{R}$, and then—putting these two together—corollaries of our general topological theorems in the special case of $\mathbb{R}$. This third type of theorem recovers the "Three hard theorems" from Spivak's Chapter 7.

1.1. General theorems.

Theorem 1.1. *If $X \to Y$ is continuous and $S \subseteq X$ is connected, then $f(S)$ is connected.*

Theorem 1.2. *If $X \to Y$ is continuous and $S \subseteq X$ is compact, then $f(S)$ is also compact.*

These two theorems holds for *any* topological space—the usual Euclidean topology in $\mathbb{R}$, or in $\mathbb{R}^n$, or in space-time, or on a graph with the graph-metric, the Zariski topology in algebraic geometry or number theory— any topological space at all.

1.2. Theorems that describe the Euclidean Topology on $\mathbb{R}$.

Theorem 1.3 (Theorem A). *A subset S of $\mathbb{R}$ is connected if and only if S is an interval.*

There are 9 types of these intervals (you explored them in the last homework).

Theorem 1.4 (Theorem B). *A subset S of $\mathbb{R}$ is compact if and only if S is closed and bounded.*

Examples of compact sets in $\mathbb{R}$ include closed bounded intervals $[a, b]$, finite sets of points, the set $\overline{S} = \{1/n \mid n \in \mathbb{N}\} \bigcup \{0\}$, or even the Cantor set. There are many others.

1.3. Specializing general theorems to the Euclidean topology on $\mathbb{R}$.

When we let both X and Y be the standard topological space $\mathbb{R}$ that you have known since high school, the two general theorems become:

Theorem 1.5. *(Intermediate Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous then f must take on all values between $f(a)$ and $f(b)$.*

Theorem 1.6. *(Extreme Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous, then f must achieve a maximum and minimum value on $[a, b]$.*

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The hardest part of these six theorems was proving the characterizations of connected and compactness for $\mathbb{R}$. Even if the details of the proof are a bit fuzzy in your head, you should understand that we used in a crucial way that $\mathbb{R}$ is *complete.* Everyone should at least understand the proof of the general theorems, their corollaries, and the statement of theorems A and B. Of course, students expecting an A+ should know more.

1.4. **Compactness.** Compactness is one of the most important concepts in mathematics; it is used not only in topology but in geometry, analysis, and algebra. Professor Smith uses it all the time in algebraic geometry research.

Definition 1.7. A subset S of a topological space X is compact if and only if *every* open cover has a finite subcover.

The point about compactness is that it often lets us see that *local properties hold globally.* One example concerns boundedness.

Definition 1.8. A function $f : X \to \mathbb{R}$ is bounded if and only if the image of $f(X)$ is bounded in $\mathbb{R}$. This means that there exists $a \in \mathbb{R}$ such that for all $x \in X$, $|f(x)| \leq a$ or equivalently $f(X) \subseteq [-a, a]$.

For example, the function $f(x) = sin(x)$ is bounded on the entire real line, since the image is contained in $[-1, 1]$. However, $f(x) = x$ is not bounded on $\mathbb{R}$.

Definition 1.9. A function $f : X \to \mathbb{R}$ is *locally bounded* if and only if for all $x \in X$, there exists an open neighborhood U of x such that the restriction $f_{|_U}$ is bounded.

An example of a function that is locally bounded but not (globally) bounded would be $f(x) = x$ on the real line.

Now what would happen with $f(x) = 1/x$? It is locally bounded if we consider it as a function on the domain $\mathbb{R} \setminus \{0\}$. But if we extended it to a function of all of $\mathbb{R}$, say by setting $f(0) = 0$, then it would not be locally bounded at 0. That is, it is not bounded in any neighborhood of 0. We can never extend $f(x) = 1/x$ to a function continuous at zero because this would contradict the Extreme Value Theorem.

The following theorem gives an example of the point that local properties on compact sets are global:

Theorem 1.10. *If $f : X \to \mathbb{R}$ is a locally bounded function and X is compact, then f is bounded.*

Proof. Because f is locally bounded we can say that for all $x \in X$, there exists an open U_x containing x such that $f(U_x) \subseteq [-N_x, N_x]$, for some $N_x \in \mathbb{N}$. These bounds that work on each U_x may vary with the point x. How will we get one that works for all points in X? We could try to take the largest of all the N_x, but there are infinitely many. What if there is no largest one, if they go off to infinity?

This is where compactness comes in. Note that

$$X \subseteq \bigcup_{x \in X} U_x,$$

so because X is compact, this cover has a finite subcover. So

$$X \subseteq U_{x_1} \bigcup \cdots \bigcup U_{x_T},$$

for some T. The function f is bounded on each of these U_{x_i}, say $f(U_{x_i}) \subseteq [-N_{x_i}, N_{x_i}]$. But now since there are only finitely many, we can let $N = \max\{N_{x_1}, ..., N_{x_T}\}$. In this case, $f(X) \subseteq [-N, N]$. Therefore, f is bounded. □

The *proof* of this theorem is very important. It shows how compactness gets used to show that *local behavior* becomes *global behavior* on a compact set. Another example of this is *uniform continuity* which is a "global form" of usual continuity, a local condition. When reading the book, try to keep in mind the idea of compactness and how it might help prove the following theorem: A continuous function $[a, b] \to \mathbb{R}$ is uniformly continuous.