PROFESSOR SMITH MATH 295 LECTURE NOTES

BY JOHN HOLLER

1. Uniform Continuity (With Compactness) and First steps in Integration

Announcements: Read Chapter 13 - up to pg. 262 by Monday and the end by Tuesday. We will be following the book pretty closely for a while, so please read the book.

1.1. Uniform Continuity.

Fix $\epsilon > 0$. For uniform continuity, we need to find $\delta > 0$ s.t. $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$ for all points a in the domain of f. This differs from the usual (local) notion of continuity, because there, we are allowed to fix the point a in the domain and then find δ that works for that a. (That δ may not work at a different point b even if the function is continuous at b—you may have to shrink δ further.) In uniform continuity, we fix the ϵ and find the *same* δ which works to check continuity at all points of the domain.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if $\forall \epsilon > 0 \exists \delta > 0 \; s.t. \; \forall x, y \in \mathbb{R}$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

In the book (Appendix to Chapter 8), it is shown that the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. We need a continually smaller δ to work for the same ϵ as we test continuity for larger and larger elements a in the domain.

Example 1.1. An example of a function that is uniformly continuous on the whole real line is $f(x) = 2x + 4$. No matter a we consider, we can take $\delta = \frac{\epsilon}{2}$, and check that the entire interval $(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})$ is mapped into $(f(a) - \epsilon, f(a) + \epsilon)$ under f.

A function doesn't have to be constant or linear to be uniformly continuous; Nick gave the example of the function $f(x) = sin\ x$. The reason it is hard to find uniformly continuous functions on the entire real line is because the domain is so large. On more compact domains, it is easy, as the following theorem shows.

Theorem 1.1. *If $f : [a,b] \to \mathbb{R}$ is continuous then f is uniformly continuous.*

The point in this theorem is that the domain $[a, b]$ is compact. The theorem does not hold on non-compact domains. For example, the function $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ but not uniformly continuous: if we fix ϵ, we will need to take smaller and smaller δ to show that $f(x)$ is continuous at very small positive values of the domain.

The theorem is proved quite nicely in the book. But the proof can be viewed through the ideas of compactness, and is a typical example of how mathematicians argue using

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compactness that local behavior determines global behavior on compact sets. So we would like to give a different way to frame the proof in the book, using the idea of compactness.

Proof. Fix $\epsilon > 0$. Since f is continuous at $y \in$ [a,b], $\exists \delta_y > 0$ s.t.

$$f(B_{\delta_y}(y)) \to B_\epsilon(f(y)).$$

We want a δ that works for all these balls. Suppose there was a $\delta > 0$ which satisfies $\delta \leq \delta_y$ for all y. Then we would be done: the δ-ball of y is a subset of the δ_y-ball, so of course it is also taken into $B_\epsilon(f(y))$.

Can't we just let $\delta = min\{\delta_y\}$? The problem is there are infinitely many δ_y, so there may not be a minimum. How about letting $\delta = inf\{\delta_y\}$? This is promising since this is a non-empty bounded below subset of $\mathbb{R}$, so the infimum exists. However, it might be zero. In fact, this is exactly what can and does happen—we might need smaller and smaller values of δ_y.

We get around this difficulty by using the remarkable idea of **compactness.** Because $[a, b]$ is compact, the open cover

$$[a, b] \subseteq \bigcup B_{\delta_y}(y)$$

must have a finite subcover, so that

$$[a, b] \subseteq B_{\delta_{y_1}}(y_1) \bigcup \cdots \bigcup B_{\delta_{y_T}}(y_T)$$

for some finitely many points $y_1 < \ldots y_T$ in $[a, b]$. In other words

$$[a, b] \subseteq (y_1 - \delta_1, y_1 + \delta_1,) \bigcup \cdots \bigcup (y_T - \delta_T, y_T + \delta_T).$$

The idea is that now we can take a minimum over the δ_i, since there are only finitely many. However, there are some little technicalities to deal with to finish the proof. These are essentially the same as in the book. The problem is that if we take δ to be the minimum of these δ_i, we we will get the required condition on δ-balls around each y_i, but there might be a δ-ball around some other point z which does not lie entirely in one of the $B_{\delta_i}(y_i)$. For this, we use the following lemma:

Lemma 1.2. *If $[a, b] \subseteq (y_1 - \delta_1, y_1 + \delta_1,) \bigcup \cdots \bigcup (y_T - \delta_T, y_T + \delta_T)$, then there exists $\delta > 0$ such that for all $z \in [a, b]$, there is some $i = 1, 2, \ldots, T$, such that $B_\delta(z) \subset B_{\delta_i}(y_i)$.*

We leave the proof of this lemma as an exercise. We also leave the proof of the theorem using it as a (not completely trivial) exercise. It is essentially in the book as well. □

Anthony asked how can $f(x) = x^2$ be uniformly continuous on the domain $[0, 1]$ if it is not uniformly continuous on the real line? The point is, that uniform continuity very much depends on the domain. The theorem tells us the function $f(x) = x^2$ is uniformly continuous on any closed bound interval, but not on the unbounded set $\mathbb{R}$.

1.2. **Integration.** The idea of integration is "area under the curve." Let f :[a,b] $\to \mathbb{R}$ be a bounded function (that is not necessarily continuous). Assume for simplicity that $f(x) \geq 0 \forall x \in [a, b]$.

The standard notation we will use for an integral is $\int_a^b f$ which is not the same notation from high school, where one usually writes $\int_a^b f(x)dx$. This denotes the area under the graph of a function. We need to make this precise.

Definition 1.3. A <u>Partition</u> of $[a, b]$ is a finite ordered subset of $[a, b]$ where a and b are the minimal and maximal elements respectively. We write P as $\{a = y_0 < y_1 < \cdots < y_n = b\}$.

<u>Note:</u> It is important to point out that for all of our definitions rely only on the assumption that the area of a rectangle is $A = bh$.

Definition 1.4. The Upper Darboux Sum with respect to P of the function f is

$$U(f, P) = \sum_{i=0}^{n} (y_i - y_{i-1})(sup\{f(x) \mid x \in [y_i, y_{i-1}]\}).$$

The Lower Darboux Sum with respect to P of f is

$$L(f, P) = \sum_{i=0}^{n} (y_i - y_{i-1})(inf\{f(x) \mid x \in [y_i, y_{i-1}]\}).$$

Why do the supremum and infemum exist in this defintion? The set is bounded since it is a subset of $f([y_i, y_{i-1}])$ which is also bounded because f is a bounded function. It is nonempty since f is a function, so the elements y_i must have some values.

Note that $U(f, P)$ overestimates the area under f and $L(f, P)$ underestimates it. In particular,

$$L(f, P) \leq U(f, P).$$

We will eventually say that the function is *integrable* if they approach one another as we refine P to finer and finer partitions that give better and better approximations of the area under the curve. You will read in the book, and we will make this precise on monday.

Someone asked what an example would be of a function that was not integrable. Professor Smith gave: $f(x) = 1$ if x irrational and 0 if x rational. Here sup=1 on any interval, and inf=0. So, on the interval $[0, 1]$ say, the upper sum with respect to any partition is 1 and the lower sum is 0. This function will not be Darboux integrable. Mathematicians have invented other types of integration theory to better handle functions like this. What do you think the "area under the curve" should be in this case? Do you feel it doesn't exist, or since "most" real numbers are irrational, should it be maybe 1? This latter approach is the subject of *Lesbeque integration* which our 395 comrades are studying right now.