Professor Smith Math 295 Lecture Notes

by John Holler

Fall 2010

1 October 25: Subspace Topology and Continuous Maps

1.1 From Last lecture

Because it is so important, we repeat:
Definition: A Topological Space is a non-empty set X (also sometimes denoted (X, $\mathcal{U}_X$)) together with a choice of a collection of subsets ($\mathcal{U}_X$) called open sets satisfying:

1. Both X and $\emptyset$ are open;
2. The collection $\mathcal{U}_X$ of open sets is closed under arbitrary unions. This means: if U_λ is open $\forall \lambda$ in some indexing set Λ then $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open
3. The collection $\mathcal{U}_X$ is closed under finite intersections. This means: if $U_1, \ldots, U_n$ are open, so is their intersection $\bigcap_{i=1}^{n} U_i$ open.

Main example (for 295): $\mathbb{R}$ with its usual Euclidean topology.
For this topology, recall that U is open $\iff$ $\forall a \in U$ $\exists \epsilon > 0$ such that the open ball $B_\epsilon(a)$ is completely contained in U.

Similarly, the main example of mathematics might be $\mathbb{R}^N$ with its usual Euclidean topology. The definition of open sets is the same as in $\mathbb{R}$, where the open balls are defined using the metric (distance) in $\mathbb{R}^N$ defined Friday.

Don't forget that a set X (including $\mathbb{R}$) can usually be topologized in many different ways! Of course, some of these ways may be more natural or important than others. For today's lecture will we be thinking mainly of the standard euclidean topology on $\mathbb{R}$. If we mention "open set" in $\mathbb{R}$ without specifying the topology, it *always* means "open in the euclidean topology." The same is generally true when discussing mathematics with anyone, though in some contexts there are other interesting topologies that get used, and is is always fair game to interrupt with the question "what topology are you using?" if it is not clear. We'll soon meet the amazing Zariski topology which allows number theorists use topology to study prime numbers, and for example plays a role in the proof of Fermat's last theorem.

Definition: Let W be a subset of a topological space. We say W is closed $\iff$ the compliment of W is open.

Example 1: $[0, \infty)$ is closed (in the Euclidean topology on $\mathbb{R}$) because its complement is $(-\infty, 0)$ which is open.

Example 2: Consider $S = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$

Prop 1: S is not open.

Proof 1: Let $a = 1$. $\forall \epsilon > 0$ $B_\epsilon(1) \nsubseteq S$

Proof 2: We have already proved there is always an irrational number between two real numbers. Any ball of any radius around 1 will contain irrational numbers, so cannot be contained in S.

Prop 2: S is not closed.

Proof: We nts $\mathbb{R} - S$ is not open. So, take $a = 0$ and fix any $B_\epsilon(0)$. By the Archimedian Property $\forall \epsilon > 0$ $\exists n \in \mathbb{N}$ $s.t.$ $0 < 1/n < \epsilon$. This number $\frac{1}{n}$ is in $B_\epsilon(0)$ and also in S. So, $\forall B_\epsilon(0)$, $B_\epsilon(0) \nsubseteq \mathbb{R} - S$

How could we make it closed? If we defined S so that it contained 0 then $0 \notin \mathbb{R} - S$. This would make the above proof not valid. In fact, the set $S \cup \{0\}$ is closed. You will prove something like this on Homework Set 7.

Intuitively, open sets are those that do no not contain *any* of their "boundary points," where as closed sets are those that contain *all* their boundary, or "limiting," points. For example the open interval $(0, 1)$ is an open set in $\mathbb{R}$, where as the set $[0, 1]$ is closed because it contains both the boundary (or "limiting", or "accumulation") points 0 and 1. Likewise the set S above is not closed because the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are accumulating to the point 0 which is not in S. Of course, some sets are neither open nor closed, such as S or the half-open interval $(0, 1]$. *Please read Handout 6 on Closed Sets where the notion of "accumulation point" is made precise.*

1.2 SUBSPACE TOPOLOGY

Let X be any topological space and let S be any nonempty subset of X. The set S has a naturally induced topology, whose open sets $U \subseteq S$ are simply those of the form $\mathcal{U} \bigcap \mathcal{S}$ where $\mathcal{U} \subseteq \mathcal{X}$ is open. We say U is "open relative to S" in this case.

Def: If S is a non-empty subset of a topological space X, then the subspace topology on S is the collection of subsets of S of the form $\mathcal{U} \bigcap \mathcal{S}$ where $\mathcal{U}$ is open in X.
Prop 3: This is a topology on S.
Proof:

1. $S = S \bigcap X$, $\emptyset = S \bigcap \emptyset$, so both S and $\emptyset$ are open in this sense.
2. Let U_λ be an arbitrary collection of open sets in S. This means each $U_\lambda = \mathcal{U}_\lambda \bigcap \mathcal{S}$ where $\mathcal{U}_\lambda$ is open in X. But $\bigcup_{\lambda \in \Lambda} U_\lambda = \bigcup_{\lambda \in \Lambda} (\mathcal{U}_\lambda \bigcap \mathcal{S}) = (\bigcup_{\lambda \in *} \mathcal{U}_\lambda) \bigcap \mathcal{S}$.
3. The proof that a finite intersection of sets open relative to S is open relative to S is similar and is left to the reader.

An example of this sort of situation is as follows: Let $X = \mathbb{R}$ and define S by $S = (0, 10]$. Then the intersection of an open set of the reals and S will be open relative to S. For example, $(9, 10]$ is open in the subspace topology on $(0, 10]$. That is, $(9, 10]$ is open relative to $(0, 10]$. BUT CAUTION: of course $(9, 10]$ is not open in $\mathbb{R}$ itself!

1.3 CONTINUOUS MAP

Def: A continuous map between topological spaces $f : X \longrightarrow Y$ is a function such that $\forall\ U \subset Y$ open, $f^{-1}(U) \subseteq X$ is open in X.

Of course if X and Y are just the real line with the usual Euclidean topology, then we have proved already that this notion of continuous is equivalent to the limit definition (for all a the limit of $f(x)$ as approaches a is $f(a)$) or the $\delta - \epsilon$ definition. However, this more abstract definition works for *every* topological space— like the double torus Professor Smith drew in class, or like space-time in physics. We can also use this to make rigorous some details Spivak brushed under the rug: A function $f : A \longrightarrow B$ for $A, B \subset \mathbb{R}$ is continuous if and only if f is continuous in the subspace topologies on A and B. You will show on the homework, for example, that if A is a closed interval, for example, say $[0, 1]$, then continuity means continuous at all points of $(0, 1)$, continuous from the left at 1 and continuous from the right at 0.