Professor Smith Math 295 Lecture Notes

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1 October 26: Continuous Mapping and Connectedness

1.1 Continuous Mapping

Definition: A mapping $f: X \longrightarrow Y$ is continuous if $\forall U \subseteq Y$ open we have that $f^{-1}(U)$ is open in X.

Example 1: We can map the set of people in this class to $\mathbb{R}$ by sending each student to their birth year. This would be a continuous mapping if we used the discrete topology on the people in the room and Euclidean topology on the range. This is because the preimage of *any set* is open, since by definition, *all* sets are open in the discrete topology. Therefore the preimage of the open sets would obviously be open.

Example 2: $X = Y = \mathbb{R}$, using the normal Euclidean topology. We proved last week that continuity (as defined above) is equivalent to saying $\lim_{x\to a} f(x) = f(a)\ \forall a \in \mathbb{R}$.

Proposition: Let A, B be subsets of $\mathbb{R}$ with $A = [a, b] \subseteq \mathbb{R}$. A function $f: A \longrightarrow B$ is continuous (meaning, using subspace topology on A, B, the preimage of an open set of B is open in A) if and only if we have $\lim_{x\to y} f(x) = f(y)\ \forall y \in (a, b)$ and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

The proof of this follows easily along the same lines as our proof when both A and B are $\mathbb{R}$. You'll do it in the next homework assignment.

Theorem: The composition of continuous mappings is continuous. This is to say that if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous mappings of topological spaces, then $f \circ g : X \longrightarrow Z$ is continuous.

In Homework 6, you are struggling to prove this with δ and ϵ in the (very!) special case where $X = Y = Z = \mathbb{R}$. This should help you appreciate the beauty and simplicity of the more abstract point of view, which proves a much more general result with a simpler proof.

Proof: We need to show that our mapping $X \to Y \to Z$ will produce an open set as the preimage of an open set of Z. So, take $U \subseteq Z$ open. We need to show that $(g \circ f)^{-1}(U)$ is open. It is clear that if U is open then $g^{-1}(U)$ is open since g is continuous. Similarly since f is continuous the preimage of the open set $g^{-1}(U)$ is open; this set is written $f^{-1}(g^{-1}(U))$. Finally, you should check $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$; this is just a general statement following immediately from the definition of the preimage. QED.

Before going any further we should clear a few things up. First we can't define continuous without the notion of a topological space. So discussions about continuity only occur about topological spaces. Also: it doesn't make sense, in complete generality, to ask whether the product of two continuous maps is continuous. We need to have a way to first talk about the *product* of two functions. When we showed that the product fg is continuous if f and g are continuous, we were dealing with functions from $\mathbb{R}$ to $\mathbb{R}$. We *defined* multiplication of two functions by taking the product of their output values. It only made sense because our range ($\mathbb{R}$) has a multiplication on it. Since not all topological spaces are fields this is not necessarily the case (such as with the double torus). Some students thought $\mathbb{R}^3$ has a multiplication on it: but what is it? Professor Smith claims there is no obvious multiplication satisfying the basic axioms we'd like.[1] On the other hand, there is a way to create a multiplication on $\mathbb{R}^2$. We can identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ (by sending $(a, b) \mapsto a + bi$) and then perform multiplication in $\mathbb{C}$.

1.2 Connectedness

Definition: A topological space X is connected if there do not exist two open nonempty sets A and B such that $X = A \bigcup B$ and $\emptyset = A \bigcap B$.

In other words, connected means that the space can't be separated into two open sets in a non-trivial way.

Example 3: Let $X = \{x \mid x^2 > 2\} \subseteq \mathbb{R}$ be the topological space given by the subspace topology from the Euclidean topology on $\mathbb{R}$. Note that

$$X = (\infty, -\sqrt{2}) \bigcup (\sqrt{2}, \infty).$$

[1] Extra credit will be given for those who can create a multiplication on $\mathbb{R}^3$ or prove that it cannot have one.

Since both of these sets are open and nonempty, and they do not overlap, X is *not* connected. Note that this would also work if we defined X so that it included $\sqrt{2}$ and $-\sqrt{2}$ since although these sets would no longer be open in $\mathbb{R}$, they would be open in the subspace topology on X.

Another important example is given by the following theorem:

Theorem: $\mathbb{R}$ is connected.

Proof: Suppose $\mathbb{R}$ is not connected. Then we can write $\mathbb{R} = A \bigcup B$ where A and B are both open, non-empty, and $A \bigcap B = \emptyset$. Now fix $a \in A$ and $b \in B$. Without loss of generality, we may say $a < b$.

Now define a set $C = \{x \in \mathbb{R} \mid [a, x] \subseteq A\}$. This is nonempty since $a \in C$. Also C is bounded above by b (otherwise A and B overlap). Because of the Completeness Axiom of $\mathbb{R}$, C has a least upper bound: we will call it τ. Of course, since together A and B cover all of $\mathbb{R}$, τ must be in either A or B.

Claim 1: $\tau \notin A$. If it were, then because A is open, there would be some (possibly really small) open ball $B_\epsilon(\tau) \subseteq A$. But then, for any y in that ball, say some y such that $\tau < y < \tau + \epsilon$, we have that $[a, y] \subseteq A$. This puts y also in C. But τ was supposed to be an upper bound for C! This contradiction guarantees that $\tau \notin A$.

Claim 2: $\tau \notin B$. This is similar, but uses that B is open. You should do it to make sure you understand!!

Therefore, since $\tau \notin A$ and $\tau \notin B$, we have a contradiction to the assumption that $A \bigcup B = \mathbb{R}$. This shows $\mathbb{R}$ is connected. QED.

Example 5: Is $\mathbb{Q}$ connected? If we used the subspace topology from the Euclidean topology on $\mathbb{R}$, the answer is NO! For instance, define

$$A = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}, \;\; B = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}.$$

Since $A = (\sqrt{2}, \infty) \bigcap \mathbb{Q}$ and $B = (-\infty, \sqrt{2}) \bigcap \mathbb{Q}$, both these sets are open in the subspace topology on $\mathbb{Q}$. Since both A and B are non-empty open sets which do not overlap and together cover $\mathbb{Q}$, we conclude that $\mathbb{Q}$ is not connected. Note that there would have been other ways to do this: the decomposition of $\mathbb{Q}$ into two open sets is not unique!