Professor Smith Math 295 Lecture Notes

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1 October 27: Proving the Intermediate Value Theorem.

1.1 From previous discussions

Since we are still getting used to ideas concerning topological spaces we repeat:

1. **Definition:** A function $f : X \to Y$ between topological spaces is continuous if the preimage of every open set of Y is open in X.

2. **Definition:** A topological space X is disconnected if there are two non-overlapping non-empty open sets which cover X. We say X is connected if it is not disconnected.

    - Ex. 1: The space $X = (-\infty, -1] \bigcup [1, \infty) \subseteq \mathbb{R}$ is disconnected since, using the subspace topology we have $A = (-\infty, -1]$ and $B = [1, \infty)$ where A and B are both open in X. They are also non-empty since $-1 \in A$ and $1 \in B$. So since $A \bigcup B = X$, we conclude that X is disconnected.
    - Ex. 2: An example of a connected topological space would be $\mathbb{R}$ which we proved in class. The point of this proof was the completeness axiom of $\mathbb{R}$. In contrast, $\mathbb{Q}$ *is disconnected*. Of course, $\mathbb{Q}$ does not satisfy the completeness axiom.
    - Ex. 3: The same proof we used to show $\mathbb{R}$ is connected can be adapted to show *any interval* in $\mathbb{R}$ is connected. These intervals are the same as in number 5 on homework 6.

We all know what intervals are from high school (and we studied the nine different types on homework 6). Rather than listing them all out, here is a succinct way to define an interval:

Definition: An interval in $\mathbb{R}$ is any subset S satisfying: if $s \in S$ and $t \in S$ with $s < t$, then for all $y \in \mathbb{R}$ such that $s < y < t$, we have $y \in S$.

Exercise: Show that if S is a subset satisfying this definition, then S is one of the nine types of intervals from Homework 6, or S is all of $\mathbb{R}$ (which we could say is $(-\infty, \infty)$), so the tenth type).

1.2 Subspaces of $\mathbb{R}$ and their connectedness

Professor Smith posed the question "Are there subsets of $\mathbb{R}$ that are connected but not one of the 9 intervals discussed?" This was answered by the next theorem.

Theorem: The only connected subspaces of $\mathbb{R}$ are the intervals.
proof: Let $X \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ that is not an interval. By definition, this means $\exists\, x_1, x_2, \tau \in \mathbb{R}\ s.t.\ x_1, x_2 \in X\ but\ \tau \notin X\ and\ x_1 < \tau < x_2$. Intuitively, there is a "hole" in X. Let

$$A = (-\infty, \tau) \bigcap X, \quad B = (\tau, \infty) \bigcap X.$$

Both A and B are open relative to X, and non-empty (since $x_1 \in A$ and $x_2 \in B$). But also $A \bigcap B = \emptyset$ and $A \bigcup B = X$ (since τ is not in X anyways). This shows X is disconnected. QED.

1.3 The Intermediate Value Theorem

Remember that our intuition of continuous functions in the real line is that a continuous function is one with no sudden breaks, holes or wild oscillations. The idea of "no sudden breaks" is that if we have a nice connected set, the function can't rip it in two, sending part of it to one place and some other part far away. That is, a continuous function ought to send connected sets to connected sets. This same intuition is valid for all topological spaces!

Theorem: If $f : X \to Y$ is continuous, and $S \subseteq X$ is a connected subset (connected under the subspace topology), then $f(S)$ is connected.

In English this means that a continuous map cannot split S and map it into a disconnected subset of Y. As DJ pointed out, this is a generalized version of the Intermediate Value Theorem.

Corollary: (The Intermediate Value Theorem) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(a) > y > f(b)$ then $\exists c \in (a, b)$ s.t. $f(c) = y$.

Proof of Corollary assuming Theorem: Let $X = Y = \mathbb{R}$ from the theorem. Take $S \subseteq \mathbb{R}$ to be $[a, b]$. This is connected so the Theorem implies that $f([a, b])$ is connected, so it must be an interval in $\mathbb{R}$. Since this interval contains $f(a)$ and $f(b)$ by definition of interval it must contain all y s.t. $f(a) < y < f(b)$. So $\exists c \in [a, b]$ s.t. $f(c) = y$. But $c \in (a, b)$, because if $a = c$ then $f(a) = f(c) = y$ contrary to our assumption that $f(a) < y$ (and similarly, $c \neq b$). QED.

Anthony asked how we can have a connected subset without ordering? Professor Smith provided the example of the set of people in our class. While there is no ordering of the set we may still define a topology on it and as a result have a notion of connectedness.

Proof of Theorem: Assume, on the contrary, that $f(S)$ is not connected (thinking of $f(S) \subseteq Y$ and using the subspace topology). So $f(S)$ can be decomposed into two non-overlapping, nonempty, open sets in $f(S)$. By definition of subspace topology, these two sets can be written

$$A = \tilde{A} \bigcap f(S), \quad B = \tilde{B} \bigcap f(S)$$

for some open sets $\tilde{A}$ and $\tilde{B}$ of Y. So we have the decomposition of $f(S)$ into the disjoint open components

$$(\tilde{A} \bigcap f(S)) \bigcup (\tilde{B} \bigcap f(S)) = f(S), \quad (\tilde{A} \bigcap f(S)) \bigcap (\tilde{B} \bigcap f(S)) = \emptyset.$$

We claim that the two sets

$$f^{-1}(\tilde{A}) \cap S \text{ and } f^{-1}(\tilde{B}) \cap S$$

actually give a decomposition of S into two non-overlapping non-empty sets of S. This would contradict our hypothesis that S is connected, and therefore complete the proof of the theorem.

To prove the claim, we need to show:

1. Both $f^{-1}(\tilde{A}) \bigcap S$ and $f^{-1}(\tilde{B}) \bigcap S$ are non-empty open sets of S.
2. They cover S; that is $S = (f^{-1}(\tilde{A}) \bigcap S) \bigcup (f^{-1}(\tilde{B}) \bigcap S)$.
3. They are disjoint; that is $(f^{-1}(\tilde{A}) \bigcap S) \bigcap (f^{-1}(\tilde{B}) \bigcap S) = \emptyset$.

All of these are easy to check.

Check for (1): Since both $\tilde{A}$ and $\tilde{B}$ are open in Y, the preimages $f^{-1}(\tilde{A})$ and $f^{-1}(\tilde{B})$ are open in X, since f is continuous. By definition of subspace topology on S, the sets $f^{-1}(\tilde{A}) \bigcap S$ and $f^{-1}(\tilde{B}) \bigcap S$ are open in S. They are non-empty, because any $a \in A = \tilde{A} \cap f(S)$ will necessarily by of the form $f(s)$ for some $s \in S$. But then s is in $f^{-1}(\tilde{A}) \cap S$. Similarly, $f^{-1}(\tilde{B}) \bigcap S$ is non empty since B is non-empty.

Check for (2): To show $S = (f^{-1}(\tilde{A}) \bigcap S) \bigcup (f^{-1}(\tilde{B}) \bigcap S)$, first note that $\supset$ obviously holds. So we only need to check that if $s \in S$, then $s \in f^{-1}(\tilde{A}) \bigcup f^{-1}(\tilde{B})$. By definition of pre-image, this is the same as showing $f(s)$ is in $\tilde{A} \bigcup \tilde{B}$. But $f(s) \in f(S)$, which is contained in $\tilde{A} \bigcap \tilde{B}$, so we are done.

Check for (3): Suppose, on the contrary, that

$$x \in (f^{-1}(\tilde{A}) \bigcap S) \bigcap (f^{-1}(\tilde{B}) \bigcap S) = f^{-1}(\tilde{A}) \bigcap f^{-1}(\tilde{B}) \bigcap S.$$

Since $x \in S$, we know $f(x) \in f(S)$. Since $x \in f^{-1}(\tilde{A})$, we know $f(x) \in \tilde{A}$, and since $x \in f^{-1}(\tilde{B})$, we know $f(x) \in \tilde{B}$. This means that $f(x) \in f(S) \bigcap \tilde{A} \bigcap \tilde{B} = (\tilde{A} \bigcap f(S)) \bigcap (\tilde{B} \bigcap f(S)) = A \bigcap B$, contrary to the choice of A and B as the two non-overlapping open sets disconnecting $f(S)$.