Professor Smith Math 295 Lecture Notes

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1 October 29: Compactness, Open Covers, and Subcovers.

1.1 Reexamining Wednesday's proof

There seemed to be a bit of confusion about our proof of the generalized Intermediate Value Theorem which is:
Theorem 1: If $f : X \to Y$ is continuous, and $S \subseteq X$ is a connected subset (connected under the subspace topology), then $f(S)$ is connected.

We'll approach this problem by first proving a special case of it:
Theorem 1: If $f : X \to Y$ is continuous and surjective and X is connected, then Y is connected.
Proof of 2: Suppose not. Then Y is not connected. This by definition means $\exists$ non-empty open sets $A, B \subset Y$ such that $A \bigcup B = Y$ and $A \bigcap B = \emptyset$. Since f is continuous, both $f^{-1}(A)$ and $f^{-1}(B)$ are open. Since f is surjective, both $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty because A and B are non-empty. But the surjectivity of f also means $f^{-1}(A) \bigcup f^{-1}(B) = X$, since $A \bigcup B = Y$. Additionally we have $f^{-1}(A) \bigcap f^{-1}(B) = \emptyset$. For if not, then we would have that $\exists x \in f^{-1}(A) \bigcap f^{-1}(B)$ which implies $f(x) \in A \bigcap B = \emptyset$ which is a contradiction. But then these two non-empty open sets $f^{-1}(A)$ and $f^{-1}(B)$ disconnect X. QED

Now we want to show that Theorem 2 implies Theorem 1. This will require the help from a few lemmas, whose proofs are on Homework Set 7:

Lemma 1: If $f : X \to Y$ is continuous, and $S \subseteq X$ is any subset of X then the restriction $f|_S : S \to Y,\ x \mapsto f(x)$ is also continuous.

Lemma 2: Given any function $f : X \to Y$ of topological spaces, f is continuous $\iff$ $f : X \to f(X)$ is continuous.

For example, notice that $sin(x)$ is continuous whether we think of its range to be $\mathbb{R}$ or the interval $[-1, 1]$.

Now we will show that assuming these two lemmas, Theorem 2 $\Rightarrow$ Theorem 1.
Proof: Take any continuous mapping $f : X \to Y$ and connected $S \subseteq X$.
By Lemma 1, $f|_S : S \to Y$ is still continuous and by Lemma 2, $f|_S : S \to f(S)$ is also continuous. By definition this function is surjective and so we may apply Theorem 2 to show that $f(S)$ is connected.

1.2 Open Covers.

We start with the definition:
Definition: Let X be any topological space and $S \subseteq X$ any subset. An open cover of S is any collection of *open* sets in X whose union contains S. In math symbols we have that a collection of open sets

$$\{U_\lambda\}_{\lambda \in \Lambda}$$

is an open cover of S if and only if

$$S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda.$$

Now we will provide some examples to help understand the nature of open covers. First, let's assume $X = \mathbb{R}$ and $S = \mathbb{R}$:

- **Example:** The one-set open cover $\{\mathbb{R}\}$ covers $\mathbb{R}$
- **Example:** Let $U_n = (n, n+2)$ where $n \in \mathbb{Z}$. The collection $\{U_n\}_{n \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$. Here we would have a lot of overlap of the open sets.
- **Example:** Let $U_n = (-n, n)$ with $n \in \mathbb{N}$. The collection $\{U_n\}_{n \in \mathbb{Z}}$ is a nested open cover of $\mathbb{R}$. Here there is even more overlapping, as each set contains all the previous ones.

  Example: Now let $X = \mathbb{R}$ and $S = (0, 1)$. The two collection of two sets $\{(-\frac{1}{2}, \frac{1}{2}), (0, \frac{3}{2})\}$ is an open cover. Since it consists of only two sets, we say it is a *finite cover.*

Sometimes a collection of open sets can cover a set S with a lot of redundancy. For example, the set $S = (0, 1)$ is has an open cover

$$\{(0, \frac{3}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, 1)\},$$

but of course, we don't actually need all three of these sets in the cover to do the job. The first and the third sets in the cover also cover S, that is the collection

$$\{(0, \frac{3}{4}), (\frac{1}{4}, 1)\}$$

is a *subcover* of S. We make the following formal definition:

Definition: If $\{U_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets, a *subcollection* is any collection of fewer sets from the collection. We can write it $\{U_\lambda\}_{\lambda \in \Lambda'}$ where Λ' is some subset of the indexing set Λ. If the collection $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover S, a subcover is a subcollection $\{U_\lambda\}_{\lambda \in \Lambda'}$ which also covers S.

It is important to note that a subcover consists of "fewer" open sets U_n, not "smaller" sets. So for example, as above, we saw that the the collection $\{(0, \frac{3}{4}), (\frac{1}{4}, 1)\}$ is a subcover of the open cover $\{(0, \frac{3}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, 1)\}$ of $(0, 1)$. However, the collection $\{(0, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, 1)\}$ is not a subcover.

There are usually many different ways to cover a set S. Here are some more open covers of $S = (0, 1)$.

1. **Example:** Let $U_n = (-\frac{n}{2}, \frac{n}{2})$. Obviously

$$\bigcup_{n \in \mathbb{N}} U_n$$

contains $(0, 1)$, so it is an open cover of $(0, 1)$. But this is a very inefficient cover. The subcover consisting of just the first two sets U_1, U_2 also covers $(0, 1)$. So the open cover $\{U_n\}_{n \in \mathbb{N}}$ has a finite subcover consisting of just two open sets.

2. **Example of a cover with no finite subcover:** Let $U_n = (\frac{1}{n}, 1)$. These sets, as n ranges through the natural numbers, give an open cover of $(0, 1)$. Indeed, you can easily prove using the archimedian property of $\mathbb{R}$ that

$$\bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 1) = (0, 1).$$

This is an interesting open cover because *no finite collection* of $\{U_n\}_{n\in\mathbb{N}}$ can cover $(0,1)$. For example, $(\frac{1}{2},1)\cup(\frac{1}{3},1)\cup(\frac{1}{10},1)$ can not cover $(0,1)$, since this union is $(\frac{1}{10},1)$, and we know (from a homework assignment) that there is a real number between 0 and $\frac{1}{10}$. It is not hard to write down a formal proof that there is no finite subcollection of the collection $\{U_n\}_{n\in\mathbb{N}}$ which also covers $(0,1)$. Indeed, the union of any finite collection of sets of the form $(\frac{1}{n},1)$ will be some interval of the form $(\frac{1}{t},1)$, but we can always find real numbers between 0 and $\frac{1}{t}$.

3. **A cover with an uncountable indexing set:** For each $x\in\mathbb{R}$, let $U_x = B_{\frac{1}{1000}}(x)$. Note that
$$\bigcup_{x\in\mathbb{R}} U_x = \mathbb{R}.$$
Here, there are uncountably many sets in the cover, the indexing set is the real numbers!

   On the other hand, this cover has a countable subcover obtained by taking balls of radius $\frac{1}{1000}$ with only *rational* centers. Since
$$\bigcup_{x\in\mathbb{Q}} U_x = \mathbb{R},$$
(prove it!), this is a countable subcover. On the other, hand, it is not too hard to see that this cover has no finite subcover. (Why?)

   We can consider a similar cover of $[0,1]$. Note that
$$[0,1]\subset\bigcup_{x\in\mathbb{Q}} U_x = \mathbb{R},$$
so these sets are an open cover of $[0,1]$. However, if we let Λ be the subset of rational numbers $\frac{0}{1000},\frac{1}{1000},\frac{2}{1000},\ldots,\frac{999}{1000},\frac{1000}{1000}$, it is easy to check that also
$$\{U_x\}_{x\in\Lambda}$$
covers $[0,1]$. This is a finite subcover of our original cover.

Some sets are special with respect to covers. Some sets have the property that *every cover has a finite subcover.* Such sets are called *compact.*

1.3 Compactness:

Definition: Let S be a subset of a topological space X. We say S is compact if every open cover of S admits some finite subcover.

We will soon prove that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.