

## RESEARCH STATEMENT

KAREN E. SMITH

My mathematical research is in the general area of *algebra*, more specifically, algebraic geometry and commutative algebra.

Algebraic geometry studies geometric objects called algebraic varieties that arise as zero sets of polynomials. A parabola, defined by the polynomial equation  $y = x^2$ , is a familiar example of an algebraic variety. In general, algebraic varieties are defined by many equations in many unknowns, and can be quite complicated. Algebraic geometry underlies many applications of mathematics to industry; these applications range from coding theory, which brings us the compact disc, to spline theory, which brings us the computer graphics essential to certain medical applications and to the entertainment industry. My own research is not motivated by any particular application of algebraic geometry, but rather by the inherent elegance and beauty of the subject.

Commutative algebra studies the underlying algebraic structures, called rings, that are associated to algebraic varieties. The rings themselves are a rich topic of investigation, and are studied by commutative algebraists without particular regard for the geometry associated with them. My own research is very much at the interface of commutative algebra and algebraic geometry. I was trained as a “pure” algebraist, but quickly found my true passion to be motivated by geometric concerns.

A long term goal of my research program is to illuminate the relationship between two different mathematical traditions: algebraic techniques based on “mod  $p$ ” arithmetic, and analytic techniques such as integration and differentiation. There appears to be a mysterious synergy between these two seemingly unrelated fields. For example, there is a large and quite diverse collection of deep theorems that can be proved using  $L^2$  tools from analysis on the one hand, or by the algebraic method of reduction modulo  $p$  and iteration of the Frobenius (or  $p$ -th power) map on the other.

Over the past five years, my focus has been on investigating several specific problems in algebraic geometry where both tight closure (a characteristic  $p$  method in commutative algebra) and the  $L^2$  methods of analysis have had an impact. This has led me to some beautiful connections and some surprising new applications of tight closure to problems in algebraic geometry. Some of this work is briefly described in

the following two sections below. Very recently, there has been a turning point in my research program, which is described in a third section.

### **Singularities in Analysis and Algebra.**

Singularity theory is one area of algebraic geometry where analytic techniques and characteristic  $p$  techniques have each had an impact. One significant portion of my research program has been to establish definitive prime characteristic (“mod  $p$ ”) characterizations of certain types of singularities that were originally analytically defined.

A century ago, algebraic geometers completed the classification of algebraic surfaces up to birational equivalence. (Two varieties are birationally equivalent if they are isomorphic except possibly along some lower dimensional subvariety.) Given any smooth projective surface, one can perform a series of manageable operations (blowing up and down) to arrive at a “minimal” or “canonical” representative of the class. Thus there is always a well-defined way to determine whether or not two algebraic surfaces are the “same.”

A crowning achievement of algebraic geometry at the end of the twentieth century was an analogous classification theory for three-folds (for which Shigefumi Mori received a Fields medal). The key insight that allowed progress here was this: it is necessary to consider certain types of “mild” singularities to get a good theory of the birational classification of smooth algebraic varieties. So even algebraic geometers who are primarily interested in smooth projective varieties were suddenly confronted with the importance of certain kinds of singularities. Among these singularities are the *rational singularities* and the *log-terminal singularities*.

Simultaneously and independently, the commutative algebraists Melvin Hochster and Craig Huneke were developing the theory of tight closure, which has had a dramatic impact in commutative algebra by providing elegant new proofs for deep and difficult old theorems and by producing surprising new results. They were not motivated by the classification problem, but rather by an entirely different collection of problems in commutative algebra. But many coincidences began to suggest a connection between the concept of rational singularities and the concept of “F-rationality” arising in tight closure theory.

In [1], I showed that if the coordinate ring of an algebraic variety is F-rational, then that variety always has rational singularities. Later, Nobuo Hara proved the converse statement, so we now know that F-rationality is essentially equivalent to rational singularities [2]. Furthermore, we have discovered that indeed, many of the important singularities in the birational classification of algebraic varieties, including rational, log terminal and log canonical singularities, all admit (or are conjectured to admit) tight closure characterizations [3],[4]. The remarkable equivalence between these disparate concepts— singularities defined using resolution of singularities and convergence of integrals on the one hand, and tight closure defined by reduction to characteristic  $p$  and iteration of the Frobenius map on the other — has fascinated me and driven much of my research.

In 1999, I achieved a break-through in my attempt to discover deeper connections between singularities and tight closure. I showed that the *test ideal* is equal to the *multiplier ideal*.<sup>1</sup> It is easy to understand this result in non-technical terms. There is a certain ideal arising in tight closure, called the test ideal, which defines the non-F-regular locus. Likewise, there is an analytically defined ideal arising in birational geometry, called the multiplier ideal, which defines the non-log terminal locus [5][6]. The equivalence of F-regularity and log terminal singularities tells us that the test ideal and the multiplier ideal are related in a precise sense (they have the same radical). The much subtler fact I proved in [7] is that in fact *these are the same ideal* (in the settings where both are defined).

The equivalence of the test ideal and the multiplier ideal was an exciting development in my research program. This equivalence provides a direct and indisputable connection between a prime characteristic concept and an analytic concept. After all, the test ideal is defined by considering the  $p^{\text{th}}$  power map in a ring of characteristic  $p$ , whereas the multiplier ideal is defined by convergence of certain integrals. This result may eventually pave the way for a much deeper understanding of how the Frobenius operator interacts with analytic tools.

### **Applications of Tight Closure to Line Bundles on Algebraic Varieties.**

Problems about line bundles form another collection of topics I have approached armed with my prime characteristic tools, knowing that analytic techniques have had a great impact on this subject. My method is to associate to each line bundle a certain commutative ring of its sections (a homogeneous coordinate ring), and then exploit operations in that ring to obtain insight about the line bundle itself. The operations I use are all obtained by “reducing to characteristic  $p$ ,” which means, for example, that I solve problems about the cone defined by the equation  $x^2 + y^2 - z^2 = 0$  by studying the solutions of this equation over finite fields of  $p$  elements (that is, “mod  $p$ ”). This approach is quite different from the standard techniques of algebraic geometry for studying line bundles, such as intersection theory and vanishing theorems.

The celebrated Kodaira Vanishing theorem asserts that the higher cohomology groups of certain line bundles are always zero [8]. This theorem was originally proved using analytic techniques, and still today many of the simplest presentations of it are heavily analytic. Remarkably, it turns out that the Kodaira Vanishing theorem is a statement about tight closure. In [9], Craig Huneke and I report on a purely commutative algebraic statement (about the behavior of tight closure of certain ideals) that we show to be *equivalent* to the Kodaira Vanishing theorem. Using the tight closure formulation of Kodaira Vanishing, we are able to give a new, algebraic proof of Kodaira Vanishing for surfaces. In another work [10], I show how to prove vanishing theorems for certain kinds of varieties (for example, quotient varieties) in situations where Kodaira Vanishing does not apply.

This work on vanishing theorems grew out of my previous work on the well-known “Fujita Conjecture.” I was inspired to begin work on Fujita’s Conjecture after hearing

---

<sup>1</sup>The same result was proved independently by Nobuo Hara of Tokyo around the same time.

of Jean-Pierre Demailly's breakthroughs on this conjecture using  $L^2$  techniques [11]. Soon I had found an approach to the conjecture using tight closure which eventually led to the only known progress so far on Fujita's Conjecture for varieties defined over field of characteristic  $p$  [12].

To understand my specific contribution, first recall Fujita's Conjecture: *if  $\mathcal{L}$  is an ample line bundle on a smooth projective variety, then the adjoint bundle  $\bigwedge^d \Omega_X \otimes \mathcal{L}^d$  is globally generated*, where  $\bigwedge^d \Omega_X$  denotes the line bundle of differential  $d$ -forms on the  $d$ -dimensional variety  $X$ . A line bundle is globally generated if for each point  $p$  on  $X$  there is some section which does not vanish at the point  $p$ . This is an important property, because the globally generated line bundles are precisely those line bundles that determine well-defined maps of varieties into projective space. In [12] and [13], I showed that if  $\mathcal{L}$  itself is globally generated, then so is  $\bigwedge^d \Omega_X \otimes \mathcal{L}^d$ . For varieties defined over the complex numbers, my result can be obtained using the Kodaira Vanishing theorem, which is what ultimately led me to investigate the relationship between Kodaira Vanishing and tight closure.

### **Rings of Differential Operators.**

Another thread in my research has been the investigation of ring theoretic properties of differential operators on singular spaces, particularly in prime characteristic. I believe that differential operators may hold the key to establishing deep connections between algebra and analysis. For example, in [14], I show that differential operators act on the test ideal in a natural way, that is, one may differentiate a test element and still have a test element. In [15], Michel Van den Bergh and I prove a characteristic  $p$  version of a conjecture of Thierry Levesseur and Toby Stafford regarding the structure of rings of differential operators.

Together with my PhD students, I have delved deeper into the structure of rings of differential operators in characteristic  $p$ . For example, I am currently supervising Manuel Blickle, who will defend his thesis in 2001, as he develops a prime characteristic analog of the well-known "intersection cohomology  $D$ -module" of Brylinski and Kashiwara. In 1998 another of my students, William Traves, completed a thesis in which he proved a special case of an old conjecture of Nakai. Nakai's conjecture predicts that the only varieties for which every differential operator can be obtained as a combination of first order operators are the smooth algebraic varieties. Traves also proved a prime characteristic version of his theorem. [16]

### **New Directions.**

My discovery that the test ideal is a multiplier ideal has launched my research into a new direction. The multiplier ideal mentioned in the preceding section is really just one of a large family of multiplier ideals. During the past year, I have been engaged in program, jointly with Lawrence Ein and Rob Lazarsfeld, to develop the general theory of multiplier ideals with particular emphasis on applications to other areas of mathematics.

Of the many projects we have in progress, only one is in press. In [17], jointly coauthored with Lawrence Ein and Rob Lazarsfeld, we describe a novel application of multiplier ideals to prove an unexpected uniform bound in commutative algebra. Specifically, we show that in a regular ring of dimension  $d$ , the  $dn$ -th symbolic power of a prime ideal  $P$  is contained in the  $n$ -th power of  $P$  for all  $n$ . Previously it was known that for each given prime ideal  $P$  there is some linear exponent  $k$  such that the  $kn$ -th symbolic power of  $P$  is contained in  $P^n$  for all  $n$  [18]. What is notable in our result is that the same, easily described, bound (namely  $d$ ) works for every prime ideal in the ring. Our paper captured the attention of the commutative algebra community and before long Hochster and Huneke had found a tight closure proof of our result. The remarkable parallels between tight closure and analysis (here, multiplier ideals) have surfaced again.

We have numerous projects in progress about which I am very excited. In one, we are able to use multiplier ideals to describe the roots of the so-called Bernstein polynomial, which is an essentially analytic object. In another, we propose an interesting invariant (we call volume) associated to any valuation of a commutative ring and use it to derive new results in valuation theory (such as an “Izumi theorem” for a general valuation). These two projects are quite orthogonal to each other, aside from the fact that both use multiplier ideals in the proofs. The former should be of interest to the group of researchers working on Bernstein polynomials while the latter will be interest to the large group of researchers in valuation theory. In a third project, we use multiplier ideals to reprove (in special cases) and make effective some work of Huneke’s on the Uniform Artin-Rees theorem. This will be of interest to yet a third group, researchers in commutative algebra. We are in an exciting period of generating many new ideas that I suspect over the next few years will lead to a flurry of interesting results in a wide range of areas.

A desire to illuminate (and exploit) the connections between prime characteristic techniques in algebra and  $L^2$ -techniques in analysis has been a long-term driving force behind much of my month-to-month research. So far in my career, I have been using algebraic tools to prove theorems in areas where analytic tools have also been of great use. I have made some specific discoveries relating the algebra and analysis. To develop as a researcher, I have decided that it is time to master the analytic tools that will help me approach this problem from both ends. To achieve this goal, next semester I will be running a seminar with a group of distinguished analysts at the University of Jyväskylä (Finland) in which we will go carefully through the analytic theory of multiplier ideals. I hope to begin collaborations with analysts that will help me develop my research in the analytic direction.

## References.

- [1] Smith, Karen E., *F-rational rings have rational singularities*, American J. Math. **119** (1997) 159–180.

- [2] Hara, Nobuo, *A characterization of rational singularities in terms of injectivity of Frobenius maps*, American J. Math. **120** (1998), no. 5, 981–996.
- [3] Smith, Karen E., *Vanishing, singularities, and effective bounds via prime characteristic local algebra*; Algebraic Geometry, Santa Cruz, 1995 Proceedings of Symposia in Pure Mathematics, eds János Kollár and Robert Lazarsfeld **62.1** (1997) 289–325.
- [4] Hara, Nobuo and Kei-ichi Watanabe *F-regular and F-pure rings vs log-terminal and log-canonical singularities*, Jour. Alg. Geom (to appear)
- [5] Nadel, Alan Michael, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Annals of Mathematics **132** (1990), no. 3, 549–596.
- [6] Ein, Lawrence, *Multiplier ideals, vanishing theorems and applications*, in Algebraic geometry—Santa Cruz 1995, 203–219, Proc. Sympos. Pure Math., **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.
- [7] Smith, Karen E., *The multiplier ideal is a universal test ideal*, Communications in Algebra, Special volume in honor of Robin Hartshorne (in press).
- [8] Kodaira, Kunihiko, *On a differential-geometric method in the theory on analytic stacks*, Proc. National Academy of Science, USA **39** (1953) 1268–72.
- [9] Huneke, Craig and Smith, Karen E., *The Kodaira vanishing theorem and tight closure*, Journal für die Reine und Angewandte Mathematik **484** (1997) 127–152.
- [10] Smith, Karen E., *Globally F-regular varieties, with applications to vanishing theorems for quotients of Fano varieties*, Michigan Math Journal, special volume in honor of Bill Fulton **48** (2000)
- [11] Demailly, Jean-Pierre, *A numerical criterion for very ample line bundles* J. Diff. Geom. **37** (1993) 323–374.
- [12] Smith, Karen E., *Fujita’s freeness conjecture in terms of local cohomology* Journal of Algebraic Geometry, **6** (1997) 417–429.
- [13] Smith, Karen E., *A tight closure proof of Fujita’s Freeness conjecture for very ample line bundles*, Mathematische Annalen, **317** (2000) 285–293.
- [14] Smith, Karen E., *The D-module structure of F-split rings*, Mathematical Research Letters, **2**, (1995) 377–386.
- [15] Smith, Karen E., and Van den Bergh, Michel, *Simplicity of rings of differential operators in prime characteristic*, Proc. London Mathematical Society **3** (1997) 32–62.
- [16] Traves, William Nathaniel, *Differential Operators and Nakai’s Conjecture*, Thesis, University of Toronto, 1998.
- [17] Ein, Lawrence, Lazarsfeld, Robert, and Smith, Karen E., *Uniform Bounds and symbolic powers on smooth varieties*, Inventiones Mathematicae (in press).
- [18] Swanson, Irena, *Linear equivalence of ideal topologies*, Math. Z. **234** (2000), no. 4, 755–775.